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Column tessellations and birth-time distributions of weighted polytopes in STIT tessellations

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Zusammenfassung

Das Ziel dieser Arbeit ist die Untersuchung einiger Modelle von zufälligen Mosaiken. Zu diesem Zweck wird zunächst ein neues Modell, das kein flächentreues zufälliges Mosaik im $\mathbb{R}^3$ ist, eingeführt – das sogenannte Column Mosaik. Die räumliche Konstruktion basiert auf einem stationären zufälligen ebenen Mosaik mit konvexen polygonalen Zellen. Zu jeder polygonalen Zelle bilden wir einen unendlichen Zylinder, der senkrecht zu der Ebene, die das ebene Mosaik enthält, ist. Jeder Zylinder wird in Zellen des räumlichen Mosaiks durch Schnitte, die zu dieser Ebene parallel sind, unterteilt. Jede räumliche Zelle ist ein gerades Prisma, dessen Grundfläche kongruent zu einer Zelle des ebenen Mosaiks ist. Somit können die Merkmale des resultierenden zufälligen räumlichen Mosaiks, nämlich Intensitäten, topologische/innere Parameter und metrische Mittelwerte von Längen, Flächen und Volumen, aus geeignet gewählten Parametern des zugrundeliegenden zufälligen ebenen Mosaiks berechnet werden. Analoge Merkmale werden für Stratum Mosaike bestimmt.

Danach führen wir markierte Poissonsche Hyperebenenprozesse ein. Dieser markierte Prozess erzeugt ein entsprechendes markiertes Poissonsches Hyperebenenmosaik. Die Verteilungen der Lebenszeit und der teilenden Hyperebene eines Objekts im Prozess der markierten Poissonschen Hyperebenenprozesse werden berechnet. Außerdem untersuchen wir die Unabhängigkeit zwischen einem gewichteten Objekt und seinem entsprechenden Geburtszeit-Vektor. Eine Beziehung zwischen Poissonschen Hyperebenenprozessen und STIT Mosaiken wird ebenfalls vorgestellt.

Zum Schluss behandeln wir für $k \in \{0, \ldots, d-1\}$ die k-dimensionalen gewichteten maximalen Polytope eines STIT Mosaiks im $\mathbb{R}^d$, wobei die inneren Volumina V_j, $j \in \{0, \ldots, k\}$, die Gewichte darstellen. Ein k-dimensionales maximales Polytop des STIT Mosaiks ist der Durchschnitt von $(d-k)$ maximalen Polytopen der Dimension $(d-1)$. Im Hinblick auf die räumlich-zeitliche Konstruktion von STIT Mosaiken hat jedes dieser $(d-k)$ Polytope eine wohldefinierte zufällige Geburtszeit. Die gemeinsame Verteilung der Geburtszeiten dieser $(d-k)$ maximalen Polytope wird berechnet und verwendet, um die Wahrscheinlichkeit, dass ein gewisses typisches maximales Segment eine feste Anzahl von inneren Knoten enthält, zu bestimmen.

Abstract

The aim of this thesis is to explore some models of random tessellations. In order to do this, first, a new model of non facet-to-facet random tessellations in $\mathbb{R}^3$ is introduced– the so-called *column tessellations*. The spatial construction is based on a stationary random planar tessellation having convex polygonal cells. From each polygonal cell we form an infinite column perpendicular to the plane containing the planar tessellation. Each column is divided into cells of the spatial tessellation by cross-sections parallel to this plane. Each spatial cell is a right prism whose base facet is congruent to a cell of the planar tessellation. Thus, the features of the resulting random spatial tessellation, namely intensities, topological/interior parameters and metric mean values involving lengths, areas and volumes, can be calculated from suitably chosen parameters of the underlying random planar tessellation. The same features are determined for stratum tessellations.

Next we introduce a marked Poisson hyperplane process. This marked process generates a corresponding marked Poisson hyperplane tessellation. The distributions of the lifetime and the dividing-hyperplane of an object in the process of marked Poisson hyperplane tessellations are computed. Furthermore, we investigate the independence between a weighted object and its corresponding birth-time vector. A relationship between Poisson hyperplane tessellations and STIT tessellations is also presented.

At the end, for $k \in \{0, \ldots, d-1\}$, we consider the k-dimensional weighted maximal polytopes of a STIT tessellation in $\mathbb{R}^d$, where the intrinsic volumes V_j, $j \in \{0, \ldots, k\}$, constitute the weights. Any k-dimensional maximal polytope of the STIT tessellation is the intersection of $(d-k)$ maximal polytopes of dimension $(d-1)$. In view of spatio-temporal construction of STIT tessellations, each of these $(d-k)$ polytopes has a well-defined random birth-time. The joint distribution of the birth-times of these $(d-k)$ maximal polytopes is calculated and used to determine the probabilities that certain typical maximal segment contains a fixed number of internal vertices.

Contents

List of Symbols

γ_{X} The intensity of some (marked) point process X

Λ A locally finite, translation invariant measure on $A(d, d-1)$

λ_d The Lebesgue measure on $\mathbb{R}^d$

$\mathbb{N}$ The set of positive natural numbers $\{1, 2, \ldots\}$

$\mathbb{N}_0$ The set of natural numbers $\{0, 1, 2, \ldots\}$

$\mathbb{Q}$ The probability measure on $G(d, d-1)$ in the decomposition (12) of Λ, also called the directional distribution of $\mathrm{PHP}(t\Lambda)$

$\mathbb{Q}_{\mathsf{X}}$ The grain distribution of a stationary particle process X; the mark distribution of a stationary marked point process X; the Palm distribution of a stationary random tessellation with respect to the typical X-type object.

$\mathcal{B}$ The Borel σ-algebra of $\mathbb{R}^d$

$\mathcal{B}(E)$ The Borel σ-algebra of a locally compact space E with a countable base

$\mathcal{R}$ The probability measure on $\mathscr{S}_+^{d-1}$ defined by $\mathcal{R}(U) := \mathbb{Q}(\{u^\perp : u \in U\})$ for $U \in \mathcal{B}(\mathscr{S}_+^{d-1})$

$\mathscr{C}'$ The system of non-empty compact subsets of $\mathbb{R}^d$

$\mathscr{C}_o$ The system of non-empty compact subsets of $\mathbb{R}^d$ with circumcenters at the origin o

$\mathscr{F}'$ The system of non-empty closed subsets of $\mathbb{R}^d$

$\mathscr{N}_s$ The abbreviation of $\mathscr{N}_s(\{0\}^2 \times \mathbb{R})$

$\mathscr{N}_s(E)$ The set of simple counting measures on E

$\mathscr{P}_k^o$ The set of k-dimensional polytopes in $\mathbb{R}^d$ with circumcenter at the origin o

$\mathscr{P}_k$ The set of k-dimensional polytopes in $\mathbb{R}^d$

$\mathscr{S}_+^{d-1}$ The upper unit half sphere in $\mathbb{R}^d$

$\mathscr{T}'$ The set of all tessellations in $\mathbb{R}^2$

$\mathscr{T}$ The set of all tessellations in $\mathbb{R}^3$

$\mathrm{PHP}(t\Lambda)$ The stationary Poisson hyperplane process with intensity measure $t\Lambda$; see Example 1.1.5

$\mathrm{PHT}(t\Lambda)$ The stationary Poisson hyperplane tessellation generated by $\mathrm{PHP}(t\Lambda)$; see Example 1.2.4

$A(d, d-1)$ The set of hyperplanes in $\mathbb{R}^d$

$G(d, d-1)$ The set of $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$

V', $\mathsf{V}'[\pi]$, $\mathsf{V}'[\bar{\pi}]$, E' and Z' The set of vertices, π-vertices, non-π-vertices, edges and cells, respectively, of a stationary random planar tessellation $\mathcal{Y}'$

V, E, P and Z The set of vertices, edges, plates and cells in that order of some random spatial tessellation (in the context of this thesis: a column tessellation $\mathcal{Y}$ or a stratum tessellation $\widetilde{\mathcal{Y}}$)

$\mathsf{X}_j^{\neq}$ The set of all j-dimensional faces of all X-type objects in some random spatial tessellation (without multiplicity). Here $j < \dim(\mathsf{X}\text{-type object})$

$\mathsf{Z}_j'^{\neq}$ The set of all j-dimensional faces of all cells in $\mathcal{Y}'$ (without multiplicity). Here $j = 0$ or $j = 1$

$\alpha_{\mathsf{x}'}$ $= \sum_{\{\mathsf{z}':\mathsf{z}'\supset\mathsf{x}'\}} \rho_{\mathsf{z}'}$ where we mostly consider the cases $\mathsf{x}' = \mathsf{v}' \in \mathsf{V}'$, $\mathsf{x}' = \mathsf{e}' \in \mathsf{E}'$, $\mathsf{x}' = \mathsf{v}'[\pi] \in \mathsf{V}'[\pi]$ and $\mathsf{x}' = \mathsf{v}'[\bar{\pi}] \in \mathsf{V}'[\bar{\pi}]$; see Page 33

$\beta_{\mathsf{v}'}$ $= \sum_{\{\mathsf{z}':(\mathsf{v}',\mathsf{z}')\in b\}} \rho_{\mathsf{z}'}$ for $\mathsf{v}' \in \mathsf{V}' = \mathsf{Z}_0'^{\neq}$; see Page 33

$\beta_{\mathsf{z}_1'}$ $= \sum_{\{\mathsf{z}':(\mathsf{z}_1',\mathsf{z}')\in b\}} \rho_{\mathsf{z}'}$ for $\mathsf{z}_1' \in \mathsf{Z}_1'^{\neq}$; see Page 33

$\epsilon_{\mathsf{v}'[\pi]}$ $= \rho_{b_\pi(\mathsf{v}'[\pi])}$ for $\mathsf{v}'[\pi] \in \mathsf{V}'[\pi]$; see Page 33

$\mu^{(2)}_{\mathsf{V}'\mathsf{E}'}$ The second moment of the number of edges adjacent to the typical vertex of $\mathcal{Y}'$

$\mu_{\mathsf{V}'\mathsf{E}'}$ The mean number of edges emanating from the typical vertex of $\mathcal{Y}'$

$\nu_{\mathsf{V}'\mathsf{Z}'}$ The mean number of owner cells of the typical vertex in $\mathcal{Y}'$

$\theta_{\mathsf{e}'}$ $= \ell(\mathsf{e}')\alpha_{\mathsf{e}'}$ for $\mathsf{e}' \in \mathsf{E}'$; see Page 33

$\theta_{\mathsf{v}'}$ $= m_{\mathsf{E}'}(\mathsf{v}')\alpha_{\mathsf{v}'}$ for $\mathsf{v}' \in \mathsf{V}'$; see Page 33

$\theta_{\mathsf{z}'}$ $= a(\mathsf{z}')\rho_{\mathsf{z}'}$ for $\mathsf{z}' \in \mathsf{Z}'$; see Page 33

$m_{\mathsf{E}'}(\mathsf{v}')$ The number of emanating edges from some vertex v' of $\mathcal{Y}'$

$n_{\mathsf{Z}}(\mathsf{z}_j')$ The number of owner cells of an object $\mathsf{z}_j' \in \mathsf{Z}_j'^{\neq}$

$n_j(\mathsf{z}')$ The number of j-dimensional faces of some planar cell z' of $\mathcal{Y}'$

$\breve{\Phi}$ The marked point process defined in the proof of Proposition 2.2.1(v)

$\grave{\Phi}$ The marked point process defined in the proof of Equation (32)

$\overline{\Phi}$ The marked point process defined in the proof of Proposition 2.2.2(i)

$\vec{\Phi}$ The marked point process defined in the proof of Proposition 2.2.14(ii)

$\widehat{\Phi}$ The marked point process defined in the proof of Proposition 2.2.1(ii)

$\widetilde{\Phi}$ The marked point process defined in the proof of Proposition 2.2.1(i)

$\mathscr{M}\mathscr{P}_k^{(t)}$ The process of k-dimensional maximal polytopes of the STIT tessellation $Y(t)$

$\mathsf{F}_{k,j}^{(t)}$ The V_j-weighted typical k-dimensional face of $\mathrm{PHT}(t\Lambda)$; see Definition 3.4.1

$\mathsf{MP}_{k,j}^{(t)}$ The V_j-weighted typical k-dimensional maximal polytope of $Y(t)$; see Definition 1.4.6

$\varrho_{k,j}^{(t)}$ The density of the jth intrinsic volume of $\mathscr{M}\mathscr{P}_k^{(t)}$

$(\mathsf{F}_{k,j}^{(t)}, \beta_1(\mathsf{F}_{k,j}^{(t)}), \ldots, \beta_{d-k}(\mathsf{F}_{k,j}^{(t)}))$ The birth-time-vector marked V_j-weighted typical k-dimensional face of $\mathrm{PHT}(t\Lambda)$; see Definition 3.4.3

$(\mathsf{MP}_{k,j}^{(t)}, \beta_1(\mathsf{MP}_{k,j}^{(t)}), \ldots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$ The birth-time vector marked V_j-weighted typical k-dimensional maximal polytope of $Y(t)$; see Definition 1.4.14

$(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$ The last birth-time marked V_j-weighted typical k-dimensional maximal polytope of $Y(t)$; see Definition 1.4.10

$\beta(p)$ The birth-time of some polytope p in $\mathbb{R}^d$

$\tau(p)$ The lifetime of some polytope p in $\mathbb{R}^d$

Introduction

Random tessellations are classical structures considered in stochastic geometry. A tessellation T of $\mathbb{R}^d$ is a locally finite family of compact and convex d-dimensional polytopes which pairwise have no common interior points and whose union fills the whole space. These polytopes are the *cells* of the tessellation T. The theory of random tessellations is an active field of current mathematical research. Besides theoretical developments, there are a lot of applications of random tessellations for example in the study of the geometry of several structures in materials science, biology, geology and other sciences; see [**26, 1, 10, 28, 3**]. Two well-known models for random tessellations are the Poisson hyperplane and Poisson-Voronoi tessellations for which we refer to [**26, 30, 2, 3**]. A Poisson hyperplane tessellation in $\mathbb{R}^2$, respectively, in $\mathbb{R}^3$, is called a Poisson line tessellation and a Poisson plane tessellation in that order. Poisson line tessellations and two-dimensional Poisson-Voronoi tessellations are *side-to-side*, that is, each side of a polygonal cell coincides with a side of a neighbouring cell (two cells of a planar tessellation are called *neighbours* if their intersection is an edge of the tessellation). In the three dimensional versions, Poisson plane and Poisson-Voronoi tessellations are *facet-to-facet*, meaning that each facet of a polyhedral cell coincides with a facet of a neighbouring cell (two cells of a three-dimensional tessellation are called *neighbours* if their intersection is a plate of the tessellation). For general dimensions, the cells of Poisson hyperplane as well as Poisson-Voronoi tessellations are in a *face-to-face* position. This property means that for any two cells of the tessellation, their intersection is either empty or is a lower dimensional face of this tessellation, which is also a common face of both cells.

In recent years there has been a growing interest in d-dimensional tessellation models that do not have the face-to-face property. A first systematic study of the effects when a three-dimensional tessellation is not facet-to-facet is given in the paper of Weiss and Cowan [**38**]. A recent study presented in [**5**] looks in depth at the planar case, building on results from the 1970s when non side-to-side tessellations first attracted attention of stochastic geometers.

Tessellations of that kind arise for example by cell division. Among these models, the *iteration stable* or STIT tessellation is of particular interest in recent times because of the number of analytically available results; see [**21, 15, 33, 34, 16, 36, 4, 31, 32, 35**] and the references cited therein. In particular, as discussed in [**19**], the STIT tessellations may serve as a reference model for hierarchical spatial cell-splitting and crack formation processes in natural sciences and technology, for example to describe geological or material phenomena, or ageing processes of surfaces. Beside, geological crack and fissure structures can appropriately modeled by

STIT tessellations [**17**]. The last model might also have application in the process of biological cell division. Chapter 4 of the thesis concentrates on the study of STIT tessellations in arbitrary dimensions. In particular, we investigate the associated lower-dimensional objects of STIT tessellations based on their spatio-temporal constructions. The development of new model classes is important for further applications to random structures in materials science, geology and biology – and Chapter 2 of this thesis contributes to that aim. The thesis is organized as follows.

In the first chapter, we collect the notions and results of stochastic and integral geometry that are used in the thesis. The first section is devoted to Palm distributions, a powerful tool to define and work with typical objects in random tessellations. In Section 1.2, we define random tessellations as well as the face-to-face property of d-dimensional tessellations. The third section mentions basic notation of random tessellations and the last one gives us a detailed understanding about STIT tessellations and their associated objects. We follow the presentation of Schneider and Weil [**30**]. Another good reference is the book of Chiu, Stoyan, Kendall and Mecke [**3**].

In Chapter 2, we consider a new model of non facet-to-facet random tessellations in $\mathbb{R}^3$ whose construction is based on a stationary random planar tessellation $\mathcal{Y}'$ in a fixed plane $\mathcal{E}$ which, without loss of generality, is assumed to be the horizontal plane $\mathbb{R}^2 \times \{0\}$. The construction is introduced in Section 2.1. Namely, for each cell z' of $\mathcal{Y}'$, we construct an infinite cylindrical column based on this cell and perpendicular to $\mathcal{E}$. Moreover, we mark the circumcenter $c(\mathsf{z}')$ of the cell z' – the center of the uniquely determined smallest ball containing z' – with a real-valued positive number $\rho_{\mathsf{z}'}$. Here, conditional upon the planar tessellation $\mathcal{Y}'$, $\rho_{\mathsf{z}'}$ is a non-random function of z', for example, the area of the cell z'. Later we give some examples in which the function $\rho_{\mathsf{z}'}$ does not only depend on z' but also on some aspects of $\mathcal{Y}'$ viewed from z'. Note that, unconditionally, $\rho_{\mathsf{z}'}$ inherits some randomness from $\mathcal{Y}'$. Such a mark is created for all cells in $\mathcal{Y}'$. Now, for each planar cell z', we construct on the line going through $c(\mathsf{z}')$ perpendicular to $\mathcal{E}$ a stationary and simple point process with intensity $\rho_{\mathsf{z}'}$. To create the spatial tessellation, the column based on z' is intersected by horizontal cross-sections located at each of the random points of that column's point process, thereby dividing the column into cells. The spatial cells which arise are right prisms and their polygonal base facets (located at the cross-sections) are vertical translations of the cells of the planar tessellation $\mathcal{Y}'$. The resulting random three-dimensional tessellation $\mathcal{Y}$ is called a *column tessellation*. The intersection of the column tessellation $\mathcal{Y}$ with any fixed plane parallel to $\mathcal{E}$ is a vertical translation of $\mathcal{Y}'$ almost surely. Column tessellations could be useful to describe crack structures in geology, as for example in the Giant's Causeway of Northern Ireland (Figure 1).

If the locations of cross-sections were identical in all columns, the column tessellation $\mathcal{Y}$ reduces to the stratum model of Mecke; see [**14**], and is facet-to-facet if and only if $\mathcal{Y}'$ is side-to-side. So, in order to be innovative, we shall be introducing mechanisms such that no section which divides a column is coplanar with a section of a neighbouring column. This implies that cells in neighbouring columns (the columns based on two neighbouring cells of $\mathcal{Y}'$) will not have a common facet. Therefore, column tessellations are never facet-to-facet.

The definitions which we shall use to specify where the cross-sections intersect the columns can vary, thus giving scope for us to consider different cases – and so to construct a rich model class. This class provides a significant generalization of column constructions considered briefly in [**38**]. In Chapter 2 we investigate which parameters of the random planar tessellation are necessary to calculate characteristics of the random spatial tessellation. This is interesting, for example, when only a planar section through a spatial column tessellation can be observed. We will find that the intensities and various mean values including the topological/interior parameters as well as the mean values involving the lengths, areas and volumes of the column tessellation $\mathcal{Y}$ can be determined from some suitable parameters of the planar tessellation $\mathcal{Y}'$ and the function $\rho_{\mathbf{z}'}$. Section 2.2 presents the computations in detail. To aid comprehension throughout this chapter, we often consider the special case where the cross-sectional plates in a column have a constant separation of 1 unit. This generates a column tessellation where all the cells have height 1. We illustrate the notation and our results using this special case. In Section 2.3, three examples are discussed when the underlying random planar tessellation $\mathcal{Y}'$ is the Poisson line tessellation, the STIT tessellation or the Poisson-Voronoi tessellation. We restrict for these planar tessellations to the stationary and isotropic case. A table of important quantities for corresponding column tessellations with constant height 1 is given. The intensities and mean values of stratum tessellations are also deduced using a similar method as for column tessellations. These results for stratum tessellations presented in Section 2.4 conclude Chapter 2.

FIGURE 1. Left: Basalt columns, approximately $6-8$ metres high above ground level, divided by 'cross-sectional plates' at approximately 30 cm spacing. There are many formations like this one (at the Giant's Causeway, Northern Ireland) around the world. Middle and right: Viewed from above, examples of pentagonal and heptagonal basalt columns at the Giant's Causeway. Photos taken by Richard Cowan.

The content of Chapter 3 turns around the so-called *marked Poisson hyperplane tessellations*. First, the scaling property of Poisson hyperplane tessellations is shown in Section 3.1. Thereafter, Section 3.2 presents the way that we construct a marked

Poisson hyperplane process: We start with a Poisson process Φ in the product space $A(d,d-1)\times[0,\infty)$ which has intensity measure $\Lambda\otimes\lambda_{[0,\infty)}$. Here Λ is assumed to be a locally finite, translation invariant measure on the space $A(d,d-1)$ of all hyperplanes in $\mathbb{R}^d$ and $\lambda_{[0,\infty)}$ is the Lebesgue measure on $\mathbb{R}$ restricted to $[0,\infty)$. Each point of Φ has the form $(\mathsf{H},\beta(\mathsf{H}))$ where H is a random hyperplane and $\beta(\mathsf{H})$ is its birth-time. Now, for a fixed time $t>0$ we put $\Phi_t := \{(\mathsf{H},\beta(\mathsf{H}))\in\Phi : \beta(\mathsf{H})\le t\}$. Then Φ_t is shown to be a stationary marked Poisson hyperplane process whose unmarked process is the Poisson hyperplane process $\mathrm{PHP}(t\Lambda)$ with intensity measure $t\Lambda$. Moreover, Φ_t generates a marked Poisson hyperplane tessellation in which for $k=0,1,\dots,d-1$, every k-face p is marked with $(d-k)$ birth-times of $(d-k)$ hyperplanes whose intersection contains p. Then the mark distribution of Φ_t for fixed $t>0$ and the lifetime distribution as well as the dividing-hyperplane distribution of a random polytope p (a random k-face or a random cell) in the process of marked Poisson hyperplane tessellations generated by $(\Phi_t, t>0)$ are determined in Section 3.3. We define a birth-time marked V_j-weighted typical k-dimensional face $(\mathsf{F}^{(t)}_{k,j},\beta_1(\mathsf{F}^{(t)}_{k,j}),\dots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j}))$ of the stationary Poisson hyperplane tessellation $\mathrm{PHT}(t\Lambda)$ in Section 3.4 and show the independence of $\mathsf{F}^{(t)}_{k,j}$ from its birth-time vector $(\beta_1(\mathsf{F}^{(t)}_{k,j}),\dots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j}))$ in Section 3.5. At the end of this chapter, Section 3.6 describes relationships between STIT tessellations and Poisson hyperplane tessellations using two fundamental connections between these two tessellation models from the paper of Schreiber and Thäle [**31**]. Some examples illustrating the relationships conclude Chapter 3.

STIT tessellations are a kind of models for non-face-to-face random tessellations in general dimensions. They were introduced by Nagel and Weiß in 2005 [**21**]. Many results including intensities and mean values of STIT tessellations were obtained in their later papers for example [**22, 23**] and in the work of Thäle and Weiß [**34**]. Within a compact convex polytope $W\subset\mathbb{R}^d$ (we assume that $d\ge 2$ in this thesis) satisfying $V_d(W)>0$, the construction of this model can be described in a short way as follows. At first, W is equipped with a random lifetime. When the lifetime of W runs out, we choose a random hyperplane H, which divides W into two non-empty sub-polytopes $W\cap\mathsf{H}^+$ and $W\cap\mathsf{H}^-$, where H^+ and H^- are two closed half-spaces specified by H. A cell-splitting hyperplane piece, namely, $W\cap\mathsf{H}$, is generated. Now, the random construction continues independently and recursively in $W\cap\mathsf{H}^+$ and $W\cap\mathsf{H}^-$ until some fixed time threshold $t>0$ is reached. The outcome $Y(t,W)$ of this algorithm is a random subdivision of W into polytopes.

According to [**21**, Theorem 1], the local tessellation $Y(t,W)$ can be extended to a random tessellation $Y(t)$ in the whole space $\mathbb{R}^d$ in such a way that for any W as above, $Y(t)$ restricted to W has the same distribution as $Y(t,W)$. We call $Y(t)$ a *STIT tessellation* of $\mathbb{R}^d$. With the STIT tessellation $Y(t)$, a number of geometric objects are associated. Chapter 4 of the thesis turns around the study of these lower-dimensional objects of $Y(t)$. More particularly, we make a link between these objects themselves, their inner structures and the spatio-temporal construction of the STIT tessellation. In order to do this, first, we write $\mathscr{M}\mathscr{P}^{(t)}_{d-1}$ for the set of cell-splitting hyperplane pieces that are introduced during the recursion steps in the above algorithm until time t. More generally, for $k = 0, \ldots, d-2$ we denote by $\mathscr{M}\mathscr{P}_k^{(t)}$ the set of k-dimensional faces of members of $\mathscr{M}\mathscr{P}_{d-1}^{(t)}$. For $k = 0, \ldots, d-1$ we call $\mathscr{M}\mathscr{P}_k^{(t)}$ the process of *k-dimensional maximal polytopes* of $Y(t)$. We also consider k-dimensional weighted maximal polytopes, where the intrinsic volumes V_j, $0 \leq j \leq k$, constitute the weights. To define them, fix $k \in \{0, \ldots, d-1\}$, $j \in \{0, \ldots, k\}$ and introduce a probability measure $\mathbb{P}_{k,j}^{(t)}$ on $\mathscr{P}_k^o$ – the (measurable) space of k-dimensional polytopes in $\mathbb{R}^d$ with circumcenters at the origin o – as follows:

$$\mathbb{P}_{k,j}^{(t)}(A) := \frac{\mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p})},$$

where $A \in \mathcal{B}(\mathscr{P}_k^o)$ and $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$. Here λ_d is the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{B} := \mathcal{B}(\mathbb{R}^d)$. A random polytope with distribution $\mathbb{P}_{k,j}^{(t)}$ is called a *V_j-weighted typical k-dimensional maximal polytope* of $Y(t)$ and is denoted by $\mathsf{MP}_{k,j}^{(t)}$. For example, $\mathsf{MP}_{1,0}^{(t)}$ is the *typical maximal segment*, whereas $\mathsf{MP}_{1,1}^{(t)}$ is the *length-weighted typical maximal segment.*

Any k-dimensional maximal polytope p of $Y(t)$ is by definition the intersection of $(d-k)$ maximal polytopes of dimension $(d-1)$. In view of the spatio-temporal construction described above, each of these $(d-k)$ polytopes has a well-defined random birth-time. We denote the birth-times of these $(d-k)$ maximal polytopes by $\beta_1(\mathsf{p}), \ldots, \beta_{d-k}(\mathsf{p})$ and order them in such a way that $0 < \beta_1(\mathsf{p}) < \ldots < \beta_{d-k}(\mathsf{p}) < t$ holds almost surely. Section 4.1 describes the joint distribution of the birth-times of the k-volume-weighted typical k-dimensional maximal polytope. We generalize this result for the case of the V_j-weighted typical k-dimensional maximal polytope in Section 4.2.

THEOREM (Theorem 4.2.1). *Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, k\}$. The joint distribution of the birth-times $\beta_1(\mathsf{MP}_{k,j}^{(t)}), \ldots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)})$ of the V_j-weighted typical k-dimensional maximal polytope $\mathsf{MP}_{k,j}^{(t)}$ of the STIT tessellation $Y(t)$ has density*

$$(s_1, \ldots, s_{d-k}) \mapsto (d-j)(d-k-1)!\frac{s_{d-k}^{k-j}}{t^{d-j}}\, \mathbf{1}\{0 < s_1 < \ldots < s_{d-k} < t\}$$

with respect to the Lebesgue measure on the $(d-k)$-dimensional simplex

$$\Delta(t) = \{(r_1, \ldots, r_{d-k}) \in \mathbb{R}^{d-k} : 0 < r_1 < \ldots < r_{d-k} < t\}.$$

In particular, if $j = k$ we obtain the uniform distribution.

An application of the previous theorem is to determine the probabilities $\mathsf{p}_{1,0}(n)$ and $\mathsf{p}_{1,1}(n)$ that the typical maximal segment $\mathsf{MP}_{1,0}^{(t)}$ or the length-weighted typical maximal segment $\mathsf{MP}_{1,1}^{(t)}$ of $Y(t)$ contains exactly $n \in \{0, 1, 2, \ldots\}$ internal vertices. Section 4.3 presents this application.

THEOREM (Theorem 4.3.1). *The probabilities* $\mathsf{p}_{1,0}(n)$ *and* $\mathsf{p}_{1,1}(n)$ *are given by*

$$\mathsf{p}_{1,0}(n) = d(d-2)! \int_0^t \int_0^{s_{d-1}} \dots \int_0^{s_2} \frac{s_{d-1}^2}{t^d} \frac{(d \cdot t - 2s_{d-1} - s_{d-2} - \dots - s_1)^n}{(d \cdot t - s_{d-1} - s_{d-2} - \dots - s_1)^{n+1}} \, \mathrm{d}s_1 \dots \mathrm{d}s_{d-2} \mathrm{d}s_{d-1}$$

and $$\mathsf{p}_{1,1}(n) = (n+1)(d-1)! \int_0^t \int_0^{s_{d-1}} \dots \int_0^{s_2} \frac{s_{d-1}^2}{t^{d-1}} \frac{(d \cdot t - 2s_{d-1} - s_{d-2} - \dots - s_1)^n}{(d \cdot t - s_{d-1} - s_{d-2} - \dots - s_1)^{n+2}} \, \mathrm{d}s_1 \dots \mathrm{d}s_{d-2} \mathrm{d}s_{d-1}.$$

In the mean, the typical maximal segment has $\frac{1}{2}(d^2 - d + 2)/(d-1)$ *internal vertices in dimension* $d \geq 2$*, whereas the length-weighted typical maximal segment in space dimension* $d \geq 3$ *has* $(d^2 - 2d + 4)/(d-2)$ *(the mean is infinite if* $d = 2$*).*

Note: Most of the content of Chapter 2 is contained in the paper [**25**] with major extensions in the presentation. Most of the content of Chapter 4 is comprised in the preprint [**24**].

CHAPTER 1

Background

1.1. Palm distributions

In the thesis, the main objects that we focus on are random tessellations. In order to introduce and work with interesting objects of tessellation models, we begin with a more general concept, the so-called point processes and employ the power of Palm calculus. The theory of point processes that we present here is a special case of the theory of random measures that is treated in the book of Schneider and Weil [**30**]. Most of the results in this section can be found there. To motivate the notion of point processes, we start with some notation and definitions.

Let E be a locally compact topological space with a countable base. Denote by $\mathcal{B}(E)$ the Borel σ-algebra of E. Let $\mathscr{M}(E)$ be the set of all Borel measures η on E which are *locally finite*, i.e. $\eta(C) < \infty$ for any compact subset C of E. $\mathscr{M}(E)$ is supplied with the σ-algebra $\mathcal{M}(E)$ generated by the evaluation maps $\eta \mapsto \eta(A)$ with $A \in \mathcal{B}(E)$, $\eta \in \mathscr{M}(E)$.

Let $\mathscr{N}(E) := \{\eta \in \mathscr{M}(E) : \eta(A) \in \mathbb{N}_0 \cup \{\infty\}$ for all $A \in \mathcal{B}(E)\}$ - the set of all *counting measures* on E. As shown in [**30**, Lemma 3.1.2], $\mathscr{N}(E)$ is a measurable subset of $\mathscr{M}(E)$. We equip $\mathscr{N}(E)$ with the σ-algebra $\mathcal{N}(E)$, the trace σ-algebra of $\mathcal{M}(E)$ on $\mathscr{N}(E)$. Let $\mathscr{N}_s(E) := \{\eta \in \mathscr{N}(E) : \eta(\{x\}) \le 1$ for all $x \in E\}$ - the set of *simple counting measures* on E. Then $\mathscr{N}_s(E)$ is a measurable subset of $\mathscr{N}(E)$. We denote the induced σ-algebra by $\mathcal{N}_s(E)$.

1.1.1. Point processes.

Definition 1.1.1. By a *point process* X in E we mean a measurable map from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space $(\mathscr{N}(E), \mathcal{N}(E))$ of counting measures on E. The point process X is *simple* if $\mathsf{X} \in \mathscr{N}_s(E)$ almost surely. A point process X in $\mathbb{R}^d$ has another name: an *ordinary point process*.

For a point process X in E, write $\mathsf{X}(\omega, A)$ instead of $\mathsf{X}(\omega)(A)$ for $\omega \in \Omega$ and $A \in \mathcal{B}(E)$. If X is simple, X and its support $\operatorname{supp}\mathsf{X}$ are shown to be isomorphic by [**30**, Lemma 3.1.4]. In this case we often identify X with $\operatorname{supp}\mathsf{X}$, where $\operatorname{supp}\mathsf{X}(\omega) := \{x \in E : \mathsf{X}(\omega, \{x\}) \neq 0\} = \{x \in E : \mathsf{X}(\omega, \{x\}) = 1\}$ for $\omega \in \Omega$. Hence, the notation $\mathsf{X}(\{\mathsf{x}\}) = 1$ and $\mathsf{x} \in \mathsf{X}$ will have the same meaning.

Definition 1.1.2. The *intensity measure* of the point process X is the measure on E defined by

$$\Theta(A) := \mathbb{E}\mathsf{X}(A) \text{ for } A \in \mathcal{B}(E).$$

We denote by $\mathscr{F}' := \mathscr{F}'(\mathbb{R}^d)$, $\mathscr{C}' := \mathscr{C}'(\mathbb{R}^d)$ the system of non-empty closed, and non-empty compact subsets of $\mathbb{R}^d$, respectively.

Definition 1.1.3. Let $E = \mathbb{R}^d$ or $E = \mathscr{F}'$. Write $\mathcal{B} := \mathcal{B}(\mathbb{R}^d)$. The point process X in E is *stationary* if $\mathsf{X} \overset{\mathcal{D}}{=} \mathsf{X} + x$, where $\mathsf{X} + x$ is the translation of X by vector $x \in \mathbb{R}^d$ defined by $(\mathsf{X}+x)(B) := \mathsf{X}(B-x)$ for $B \in \mathcal{B}$ or $B \in \mathcal{B}(\mathscr{F}')$, respectively. If $B \in \mathcal{B}(\mathscr{F}')$ then $B - x := \{F - x : F \in B\}$.

The point process X is isotropic if $\mathsf{X} \overset{\mathcal{D}}{=} \vartheta\mathsf{X}$ for any rotation $\vartheta \in SO_d$. Here, $\overset{\mathcal{D}}{=}$ means equal in distribution.

Example 1.1.4. A simple point process X in E with intensity measure Θ is called a *Poisson process* in E if the two following conditions are satisfied:

(i) X has *Poisson counting variables*, that is, the random variable $\mathsf{X}(B)$ is Poisson distributed for any Borel subset B of E with $0 < \Theta(B) < \infty$. In particular,
$$\mathbb{P}(\mathsf{X}(B) = k) = e^{-\Theta(B)}\frac{\Theta(B)^k}{k!} \tag{1}$$
for $k \in \mathbb{N}_0$.

(ii) X has *independent increments*, that is, $\mathsf{X}(B_1), \ldots, \mathsf{X}(B_k)$ are independent random variables for $k \in \mathbb{N}$ and pairwise disjoint Borel subsets $B_1, \ldots, B_k$ of E.

Example 1.1.5. A *Poisson hyperplane process* in $\mathbb{R}^d$ is a Poisson process in the space $A(d, d-1)$ of hyperplanes in $\mathbb{R}^d$ and hence a Poisson process in $\mathscr{F}'$ with intensity measure concentrated on $A(d, d-1)$. Assume that we have a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity measure Θ. Then according to [**30**, Theorem 4.4.2], there are a number $\gamma \in [0, \infty)$ and a probability measure $\mathbb{Q}$ on $G(d, d-1)$ – the set of $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$ – which are uniquely determined by Θ such that
$$\int_{A(d,d-1)} f \mathrm{d}\Theta = \gamma \int_{G(d,d-1)} \int_{H_o^\perp} f(H_o + x)\lambda_{H_o^\perp}(\mathrm{d}x)\mathbb{Q}(\mathrm{d}H_o).$$
γ and $\mathbb{Q}$ are called the *intensity* and *directional distribution* of the stationary Poisson hyperplane process. A stationary Poisson hyperplane process with intensity measure Θ will be denoted by PHP(Θ). Furthermore, let $\mathscr{S}_+^{d-1}$ be the upper unit half-sphere in $\mathbb{R}^d$. For $U \in \mathcal{B}(\mathscr{S}_+^{d-1})$, put
$$\mathcal{R}(U) := \mathbb{Q}(\{u^\perp : u \in U\}),$$
where $u^\perp$ denotes the orthogonal complement of the linear subspace spanned by u. Then a stationary Poisson hyperplane process is non-degenerate if $\mathcal{R}$ is not concentrated on a great half-subsphere of $\mathscr{S}_+^{d-1}$. The last comment is that, the connected component of the complement of the union of all hyperplanes belonging to a stationary Poisson hyperplane process are random open polyhedral sets. Their closures are the *cells induced by* this process.

Proposition 1.1.6. *If* X *is a stationary ordinary point process, its intensity measure* Θ *is invariant under translations.*

Proof. We have to prove that for any $x \in \mathbb{R}^d$ and $B \in \mathcal{B}$,

$$\Theta(B + x) = \Theta(B).$$

Indeed, by definition and the stationarity of X, it is easy to see that $\Theta(B+x) = \mathbb{E}\mathsf{X}(B+x) = \mathbb{E}(\mathsf{X}-x)(B) = \mathbb{E}\mathsf{X}(B) = \Theta(B)$. □

Let X be a stationary ordinary point process. Additionally assume that its intensity measure Θ is locally finite, that is, satisfies $\Theta(C) < \infty$ for any compact subset C of $\mathbb{R}^d$. From the fact that Θ is invariant under translations shown in Proposition 1.1.6, Θ must be the Lebesgue measure λ_d on $\mathbb{R}^d$ up to a constant factor, namely

$$\Theta = \gamma_{\mathsf{X}} \lambda_d \tag{2}$$

for a constant $0 \leq \gamma_{\mathsf{X}} < \infty$. We call γ_{X} the *intensity* of the stationary point process X in $\mathbb{R}^d$.

Now we mention a well-known theorem which will be used frequently.

Theorem 1.1.7 (Campbell theorem [**30**, Theorem 3.1.2])**.** *Let X be a point process in E with intensity measure Θ. Furthermore, let $f : E \to \mathbb{R}$ be a non-negative, measurable function. Then $\int_E f \mathrm{d}\mathsf{X}$ is measurable, and*

$$\mathbb{E} \int_E f \mathrm{d}\mathsf{X} = \int_E f \mathrm{d}\Theta.$$

Note that if additionally X is simple, Campbell theorem can be rewritten in the form

$$\mathbb{E} \sum_{\mathsf{x} \in \mathsf{X}} f(\mathsf{x}) = \int_E f \mathrm{d}\Theta.$$

Remark 1.1.8. If X is a stationary and simple ordinary point process then Theorem 1.1.7 and the decomposition (2) give us

$$\gamma_{\mathsf{X}} = \frac{1}{\lambda_d(B)} \mathbb{E} \sum_{\mathsf{x} \in \mathsf{X}} \mathbf{1}_B(\mathsf{x}) \tag{3}$$

for $B \in \mathcal{B}$ satisfying $0 < \lambda_d(B) < \infty$. Equation (3) supplies us another definition of γ_{X} as the mean number of points of X per unit volume. Note that the right-hand side of the equation does not depend on the choice of the Borel subset B of $\mathbb{R}^d$.

1.1.2. Grain distribution of stationary particle processes.

Definition 1.1.9. A (*simple*) *particle process* X in $\mathbb{R}^d$ is a (simple) point process in $E = \mathscr{F}'$ which concentrates on the subset $\mathscr{C}'$, that is, $\Theta(\mathscr{F}' \setminus \mathscr{C}') = 0$, where Θ is the intensity measure of X.

Stationarity for a particle process is already introduced in Definition 1.1.3. For each $C \in \mathscr{C}'$ denote by $c(C)$ its circumcenter - the center of the (uniquely determined) smallest ball containing C. Put $\mathscr{C}_o := \{C \in \mathscr{C}' : c(C) = o\}$, where o is the origin. The next theorem provides an expression for the intensity measure Θ of X.

Theorem 1.1.10 ([**30**, Equation (4.3)]). *Let* X *be a stationary particle process in* $\mathbb{R}^d$ *and* $\Theta \neq 0$ *be its intensity measure. Then there exist a number* $0 < \gamma_{\mathsf{X}} < \infty$ *and a probability measure* $\mathbb{Q}_{\mathsf{X}}$ *on* $\mathscr{C}_o$ *such that for every* Θ*-integrable function* f *on* $\mathscr{C}'$*,*

$$\int\limits_{\mathscr{C}'} f \mathrm{d}\Theta = \gamma_{\mathsf{X}} \int\limits_{\mathscr{C}_o} \int\limits_{\mathbb{R}^d} f(C+x)\lambda_d(\mathrm{d}x)\mathbb{Q}_{\mathsf{X}}(\mathrm{d}C).$$

The number γ_{X} *and the probability measure* $\mathbb{Q}_{\mathsf{X}}$ *are uniquely determined.*

We call γ_{X} the *intensity* and $\mathbb{Q}_{\mathsf{X}}$ the *grain distribution* of the stationary particle process X. For any translation invariant, measurable function $\varphi : \mathscr{C}' \to \mathbb{R}$ which is either non-negative or $\mathbb{Q}_{\mathsf{X}}$-integrable, we define the φ*-density* of X by

$$\overline{\varphi}(\mathsf{X}) := \gamma_{\mathsf{X}} \int\limits_{\mathscr{C}_o} \varphi \, \mathrm{d}\mathbb{Q}_{\mathsf{X}}.$$

The following theorem gives two different representations of the φ-density.

Theorem 1.1.11 ([**30**, Theorem 4.1.3]). *Let* X *be a stationary simple particle process in* $\mathbb{R}^d$ *with grain distribution* $\mathbb{Q}_{\mathsf{X}}$*, and let* $\varphi : \mathscr{C}' \to \mathbb{R}$ *be a translation invariant measurable function which is non-negative or* $\mathbb{Q}_{\mathsf{X}}$*-integrable.*

(a) For any $B \in \mathcal{B}$ *with* $0 < \lambda_d(B) < \infty$*,*

$$\overline{\varphi}(\mathsf{X}) = \frac{1}{\lambda_d(B)} \mathbb{E} \sum_{C \in \mathsf{X},\, c(C) \in B} \varphi(C).$$

(b) For any compact convex subset W *of* $\mathbb{R}^d$ *whose volume* $V_d(W)$ *is positive,*

$$\overline{\varphi}(\mathsf{X}) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \mathbb{E} \sum_{C \in \mathsf{X},\, C \subset rW} \varphi(C).$$

Proposition 1.1.12 ([**30**, Equation (4.8)]). *Let* X *be a stationary simple particle process with grain distribution* $\mathbb{Q}_{\mathsf{X}}$*. Then* $\mathbb{Q}_{\mathsf{X}}$ *can be expressed in two ways as follows:*

$$\mathbb{Q}_{\mathsf{X}}(A) = \frac{1}{\gamma_{\mathsf{X}} \lambda_d(B)} \mathbb{E} \sum_{\mathsf{C} \in \mathsf{X},\, c(\mathsf{C}) \in B} \mathbf{1}_A(\mathsf{C} - c(\mathsf{C})), \tag{4}$$

and

$$\mathbb{Q}_{\mathsf{X}}(A) = \frac{1}{\gamma_{\mathsf{X}}} \cdot \lim_{n \to \infty} \frac{1}{n^d} \mathbb{E} \sum_{\mathsf{C} \in \mathsf{X},\, \mathsf{C} \subset [n]} \mathbf{1}_A(\mathsf{C} - c(\mathsf{C})) \tag{5}$$

for $B \in \mathcal{B}$ *with* $0 < \lambda_d(B) < \infty$*,* $[n]$ *the centered cube in* $\mathbb{R}^d$ *with volume* n^d *and* $A \in \mathcal{B}(\mathscr{C}_o)$*.*

Proof. Apply Theorem 1.1.11 for $\varphi : \mathscr{C}' \to \mathbb{R}$ given by $\varphi(C) := \mathbf{1}_A(C - c(C))$. □

A random set with distribution $\mathbb{Q}_{\mathsf{X}}$ is called the *typical grain* of X. The last comment of this subsection mentions the meaning of γ_{X}. In particular, when $A = \mathscr{C}_o$, we obtain

$$\gamma_{\mathsf{X}} = \frac{1}{\lambda_d(B)} \mathbb{E} \sum_{\mathsf{C} \in \mathsf{X},\, c(\mathsf{C}) \in B} 1 = \lim_{n \to \infty} \frac{1}{n^d} \mathbb{E} \sum_{\mathsf{C} \in \mathsf{X},\, \mathsf{C} \subset [n]} 1$$

for $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$. Taking B to be a Borel subset of $\mathbb{R}^d$ with volume 1 then the intensity of a stationary simple particle process X can be understood as the mean number of circumcenters of objects of X per unit volume. Consequently, Equations (4) and (5) can be written in the form

$$\mathbb{Q}_{\mathsf{X}}(A) = \frac{\mathbb{E} \sum\limits_{\mathsf{C} \in \mathsf{X},\, c(\mathsf{C}) \in B} \mathbf{1}_A(\mathsf{C} - c(\mathsf{C}))}{\mathbb{E} \sum\limits_{\mathsf{C} \in \mathsf{X},\, c(\mathsf{C}) \in B} 1} \tag{6}$$

and

$$\mathbb{Q}_{\mathsf{X}}(A) = \lim_{n \to \infty} \frac{\mathbb{E} \sum\limits_{\mathsf{C} \in \mathsf{X},\, \mathsf{C} \subset [n]} \mathbf{1}_A(\mathsf{C} - c(\mathsf{C}))}{\mathbb{E} \sum\limits_{\mathsf{C} \in \mathsf{X},\, \mathsf{C} \subset [n]} 1} \tag{7}$$

1.1.3. Mark distribution of marked point processes.

Definition 1.1.13. A *marked point process* in $\mathbb{R}^d$ with mark space M – a locally compact space with a countable base – is a simple point process X in $E = \mathbb{R}^d \times M$ with intensity measure Θ satisfying $\Theta(C \times M) < \infty$ for any compact subset C of $\mathbb{R}^d$.

In this subsection we concentrate on the marked point process X in $\mathbb{R}^d$ which is stationary. The latter means that for any $x \in \mathbb{R}^d$ we have $\mathsf{X} \overset{\mathcal{D}}{=} \mathsf{X} + x$, where $\mathsf{X} + x = \{(\mathsf{y} + x, \mathsf{m}) : (\mathsf{y}, \mathsf{m}) \in \mathsf{X}\}$. We introduce the so-called mark distribution of X in the next theorem.

Theorem 1.1.14 ([**30**, Theorem 3.5.1])**.** *If X is a stationary marked point process in $\mathbb{R}^d$ with mark space M and intensity measure $\Theta \neq 0$, then Θ possesses the decomposition*

$$\Theta = \gamma_{\mathsf{X}} \lambda_d \otimes \mathbb{Q}_{\mathsf{X}}$$

with a number $0 < \gamma_{\mathsf{X}} < \infty$ and a probability measure $\mathbb{Q}_{\mathsf{X}}$ on M. Here $\mathbb{Q}_{\mathsf{X}}$ is uniquely determined.

γ_{X} and $\mathbb{Q}_{\mathsf{X}}$ are called the *intensity* and the *mark distribution* of the stationary marked point process X. The next theorem is a direct consequence of Theorems 1.1.7 and 1.1.14.

Theorem 1.1.15. *Let X be a stationary marked point process in $\mathbb{R}^d$ with mark space M and intensity measure $\Theta \neq 0$. Let $f : \mathbb{R}^d \times M \to \mathbb{R}$ be a non-negative measurable function. Then $\sum\limits_{(\mathsf{y},\mathsf{m}) \in \mathsf{X}} f(\mathsf{y}, \mathsf{m})$ is measurable, and*

$$\mathbb{E} \sum_{(\mathsf{y},\mathsf{m}) \in \mathsf{X}} f(\mathsf{y}, \mathsf{m}) = \int\limits_{\mathbb{R}^d \times M} f(y, m) \Theta(\mathrm{d}(y, m)) = \gamma_{\mathsf{X}} \int\limits_{M} \int\limits_{\mathbb{R}^d} f(y, m) \lambda_d(\mathrm{d}y) \mathbb{Q}_{\mathsf{X}}(\mathrm{d}m).$$

In particular, for $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$ and $A \in \mathcal{B}(M)$,

$$\mathbb{Q}_{\mathsf{X}}(A) = \frac{\mathbb{E} \sum_{(\mathsf{y},\mathsf{m}) \in \mathsf{X}} \mathbf{1}_B(\mathsf{y}) \mathbf{1}_A(\mathsf{m})}{\mathbb{E} \sum_{(\mathsf{y},\mathsf{m}) \in \mathsf{X}} \mathbf{1}_B(\mathsf{y})}. \tag{8}$$

Corollary 1.1.16. *Let X be a stationary simple particle process in $\mathbb{R}^d$; see Section 1.1.2, and X' the stationary marked point process in $\mathbb{R}^d \times \mathscr{C}_o$ generated by marking $c(\mathsf{C})$ with $\mathsf{C} - c(\mathsf{C})$ for any $\mathsf{C} \in \mathsf{X}$. Then the mark distribution $\mathbb{Q}_{\mathsf{X}'}$ of the marked point process X' is equal to the grain distribution $\mathbb{Q}_{\mathsf{X}}$ of the particle process X.*

Proof. Indeed, for $A \in \mathcal{B}(\mathscr{C}_o)$, Theorem 1.1.15 and Equation (6) give us

$$\begin{aligned}
\mathbb{Q}_{\mathsf{X}'}(A) &= \frac{\mathbb{E} \sum_{(c(\mathsf{C}),\mathsf{C}-c(\mathsf{C})) \in \mathsf{X}'} \mathbf{1}_B(c(\mathsf{C})) \mathbf{1}_A(\mathsf{C} - c(\mathsf{C}))}{\mathbb{E} \sum_{(c(\mathsf{C}),\mathsf{C}-c(\mathsf{C})) \in \mathsf{X}'} \mathbf{1}_B(c(\mathsf{C}))} \\
&= \frac{\mathbb{E} \sum_{\mathsf{C} \in \mathsf{X}} \mathbf{1}_B(c(\mathsf{C})) \mathbf{1}_A(\mathsf{C} - c(\mathsf{C}))}{\mathbb{E} \sum_{\mathsf{C} \in \mathsf{X}} \mathbf{1}_B(c(\mathsf{C}))} = \mathbb{Q}_{\mathsf{X}}(A).
\end{aligned}$$

□

1.2. Ramdom tessellations

1.2.1. Definition and related notions. Write $\mathscr{P}_3$ for the set of 3-dimensional polytopes in $\mathbb{R}^3$.

Definition 1.2.1. A *tessellation* in $\mathbb{R}^3$ is a set $\{p_i, i \in \mathbb{N}\}$ satisfying the following properties:

(i) $p_i \in \mathscr{P}_3$ for all $i \in \mathbb{N}$,
(ii) $\bigcup_{i=1}^{\infty} p_i = \mathbb{R}^3$,
(iii) $\operatorname{int} p_i \cap \operatorname{int} p_j = \emptyset$ for all $i, j \in \mathbb{N}$ and $i \neq j$, where the notation $\operatorname{int} p$ signifies the interior of some polytope p,
(iv) $\{p_i, i \in \mathbb{N}\}$ is locally finite, that is, for any bounded subset B of $\mathbb{R}^3$, there exist only finitely many $i \in \mathbb{N}$ such that $p_i \cap B \neq \emptyset$.

Expressed in words, a tessellation in $\mathbb{R}^3$ is a locally finite set of 3-dimensional polytopes which have pairwise no common interior points and whose union fills the whole space. The polytopes p_i are the *cells* of the tessellation. The set of *vertices* of the tessellation is given as the set of all 0-dimensional faces of the cells (called *cell-apices*). Now if we consider the union of all 1-dimensional faces of the cells (called *cell-ridges*) then an *edge* of the tessellation is a linear segment in this union between two vertices and with no internal vertices (that is, no vertex appears in the relative interior of this segment). Analogously, a 2-dimensional convex polygon which is a subset of the union of all 2-dimensional faces of cells (called *cell-facets*),

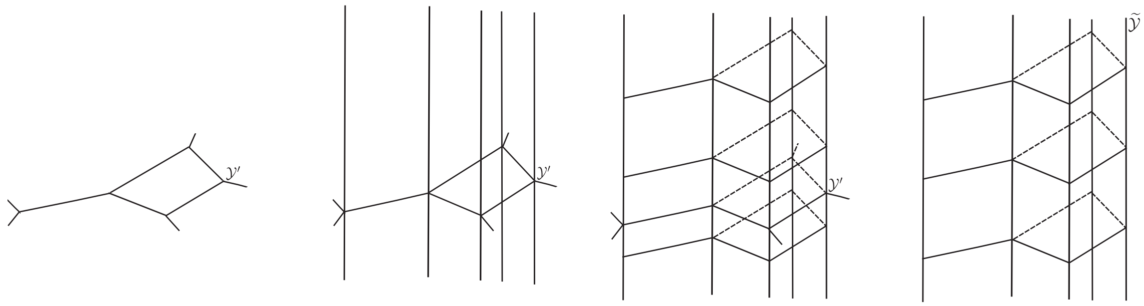

FIGURE 2. Stratum tessellation $\widetilde{\mathcal{Y}}$ with constant height 1. Here the four steps building up a stratum tessellation are shown: Starting with the planar tessellation $\mathcal{Y}'$, next the columns are given by the cells of $\mathcal{Y}'$. Then the columns with cuts are shown, whereas in the last figure $\mathcal{Y}'$ is removed.

whose boundary is contained in the union of cell-ridges and whose relative interior has no vertices and edges is called a *plate* of the tessellation.

We write $\mathscr{T}$ for the set of all tessellations in $\mathbb{R}^3$.

To prepare for the concept of random tessellations in $\mathbb{R}^3$, we need a measurable structure on $\mathscr{T}$. We equip $\mathscr{T}$ with the σ-field $\mathcal{T}$ generated by the evaluation maps $T \mapsto |T \cap A|$, where $T \in \mathscr{T}$, $A \in \mathcal{B}(\mathscr{P}_3)$ and $|\cdot|$ stands for the cardinality.

Definition 1.2.2. A random tessellation in $\mathbb{R}^3$ is a simple particle process in $\mathbb{R}^3$ which is concentrated on $(\mathscr{T}, \mathcal{T})$.

The corresponding definition of a random d-dimensional tessellation is an obvious generalization of random tessellations in $\mathbb{R}^3$. Besides, a random d-dimensional tessellation could be also defined as a random closed subset of $\mathbb{R}^d$ formed by the union of all cell-boundaries. We switch between both definitions arbitrarily.

If $d = 2$ we write $(\mathscr{T}', \mathcal{T}')$ for the measurable space of all tessellations in $\mathbb{R}^2$.

Remark 1.2.3. A stationary random tessellation is a stationary simple particle process; see the theory of particle process in Section 1.1.2.

Example 1.2.4. The system of the *cells induced by* a non-degenerate stationary Poisson hyperplane process in $\mathbb{R}^d$; see Example 1.1.5, is a stationary *Poisson hyperplane tessellation.* The stationary Poisson hyperplane tessellation generated by the stationary Poisson hyperplane process PHP(Θ) is denoted by PHT(Θ).

Example 1.2.5. Mecke [**14**] introduces a tessellation in $\mathbb{R}^3$, the so-called stratum tessellation. The following construction is a simple case of his exposition: Starting with a stationary random tessellation $\mathcal{Y}'$ on the horizontal plane $\mathcal{E} := \mathbb{R}^2 \times \{0\}$, we construct an infinite cylindrical column based on each cell of $\mathcal{Y}'$ and perpendicular to $\mathcal{E}$. Then we cut all infinite columns with planes parallel to $\mathcal{E}$. Note that the distance between $\mathcal{E}$ and its nearest plane in the upper half space is a random variable which is uniformly distributed in $[0,1)$. Moreover, the distance between two consecutive planes is always 1. The resulting stationary random spatial tessellation $\widetilde{\mathcal{Y}}$ is called a *stratum tessellation with height* 1. The intersection of $\widetilde{\mathcal{Y}}$ with any fixed plane parallel to $\mathcal{E}$ is a vertical translation of $\mathcal{Y}'$ almost surely. Any cell of $\widetilde{\mathcal{Y}}$ that has arisen is a right prism with height 1, where its base facet is a vertical translation of a cell of $\mathcal{Y}'$. We emphasize that each cell of the planar tessellation $\mathcal{Y}'$ is not a base facet of any cell of the stratum tessellation $\widetilde{\mathcal{Y}}$; see Figure 2.

1.2.2. Face-to-face property of d-dimensional tessellations.

Definition 1.2.6. In a three-dimensional tessellation T, its cells are *facet-to-facet* if and only if for any two cells $z_k, z_l \in T$ with $\dim(z_k \cap z_l) = 2$, $z_k \cap z_l$ is a common facet (2-dimensional face or 2-face for short) of both cells.

This common facet forms a plate of T. Then the tessellation T satisfying Definition 1.2.6 is said to be facet-to-facet. We write $\widehat{\mathscr{T}}$ for the set of all facet-to-facet tessellations in $\mathbb{R}^3$. Then $\widehat{\mathscr{T}}$ is a measurable subset of $\mathscr{T}$. We equipped $\widehat{\mathscr{T}}$ with the σ-algebra $\widehat{\mathcal{T}}$, the trace σ-algebra of $\mathcal{T}$ on $\widehat{\mathscr{T}}$.

Definition 1.2.7. A facet-to-facet random tessellation in $\mathbb{R}^3$ is a simple particle process in $\mathbb{R}^3$ which is concentrated on $(\widehat{\mathscr{T}}, \widehat{\mathcal{T}})$.

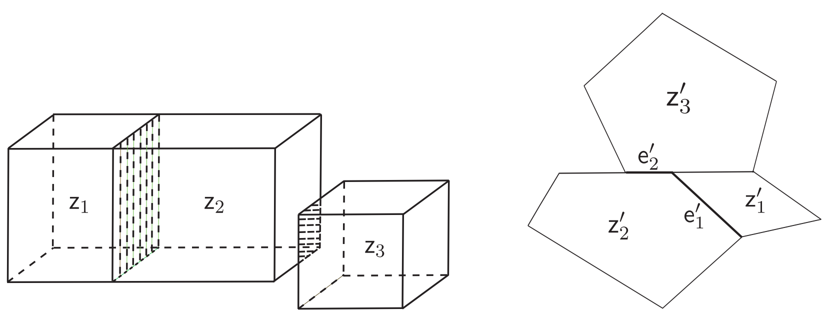

FIGURE 3. The cells z_1 and z_2 are facet-to-facet. The vertical slashed polygon is a common facet of both cells and also a plate of the spatial tessellation. In contrast, the cells z_2 and z_3 are not facet-to-facet. Their intersection (the horizontal slashed polygon) is neither a facet of z_2 nor z_3. The planar cells z'_1 and z'_2 are side-to-side. Their common side is the edge e'_1. The cells z'_2 and z'_3 are not side-to-side because their intersection – the edge e'_2 – is neither a side of z'_2 nor a side of z'_3.

Another terminology has been used by stochastic geometers, namely, face-to-face; see [**30**, Page 447].

Definition 1.2.8. Fix $d \geq 2$. The cells of a d-dimensional tessellation are said to be in a *face-to-face* position if and only if for two arbitrary cells, their intersection is either empty or a j-dimensional face of the tessellation, which is also a common j-face of these both cells, $j \in \{0, \ldots, d-1\}$.

The equivalence between facet-to-facet and face-to-face in the context of random 3-dimensional tessellations is shown in [**6**, Lemma 1]. For a planar tessellation the term *side-to-side* is used – a side-to-side planar tessellation. The latter means that for any two polygonal cells of the tessellation, their intersection is either empty, a vertex, or an edge (which is a common 1-face (called *side*) of both cells) of the tessellation. Examples of cells which are facet-to-facet (side-to-side) or not facet-to-facet (not side-to-side) are given in Figure 3.

Example 1.2.9. Poisson line tessellations are side-to-side. Poisson plane tessellations are facet-to-facet. Poisson hyperplane tessellations are face-to-face.

Example 1.2.10. The 3-dimensional stratum tessellation with constant height 1 in Example 1.2.5 is facet-to-facet if and only if the generating stationary random planar tessellation is side-to-side.

1.3. Basic notation for random tessellations

With the help of the Palm calculus, we introduce the basic notation of tessellations following [**38**]. For a stationary random 3-dimensional tessellation T, we deal with four kinds of *primitive elements*: vertices, edges, plates and cells. The corresponding sets of these primitive elements are V, E, P and Z. The primitive elements are k-dimensional polytopes, $k = 0, 1, 2, 3$, which do not have any other elements in their relative interior. An object belonging to a set X is often referred to as "an X-type object" or "an object of type X". The intensity of objects of the set X is denoted by γ_{X}. It is assumed henceforth that $0 < \gamma_{\mathsf{X}} < \infty$; this is the case for all examples considered.

We can consider the set V of vertices as a stationary simple point process in $\mathbb{R}^3$. Similarly, the sets E, P and Z can be regarded as stationary simple particle processes in $\mathbb{R}^3$. The intensity of objects of the set X ($\mathsf{X} \in \{\mathsf{V}, \mathsf{E}, \mathsf{P}, \mathsf{Z}\}$) is the mean number of points of X (when $\mathsf{X} = \mathsf{V}$) or circumcenters of X-type objects (when $\mathsf{X} \neq \mathsf{V}$) per unit volume; see Subsections 1.1.1 and 1.1.2.

Definition 1.3.1. Given $\mathsf{X}, \mathsf{Y} \in \{\mathsf{V}, \mathsf{E}, \mathsf{P}, \mathsf{Z}\}$ and $\mathsf{X} \neq \mathsf{Y}$. An object x of X is said to be *adjacent* to an object y of Y if either $\mathsf{x} \subseteq \mathsf{y}$ or $\mathsf{y} \subseteq \mathsf{x}$.

For an element $\mathsf{x} \in \mathsf{X}$ the number of Y-type objects adjacent to x is denoted by $m_{\mathsf{Y}}(\mathsf{x})$. Let μ_{XY} be the mean number of Y-type objects adjacent to the typical object of X. Formally, $\mu_{\mathsf{XY}} := \mathbb{E}_{\mathsf{X}}[m_{\mathsf{Y}}(\mathsf{x})]$, where $\mathbb{E}_{\mathsf{X}}$ denotes the expectation with respect to a probability measure $\mathbb{Q}_{\mathsf{X}}$ defined below.

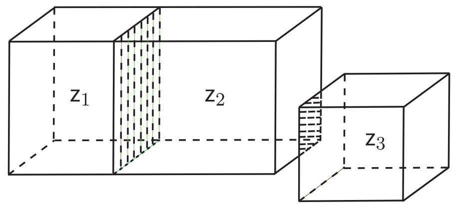

FIGURE 4. Each plate is adjacent to 2 cells, regardless the facet-to-facet property of cells. Indeed, the vertical slashed plate is adjacent to the cells z_1 and z_2 which are facet-to-facet cells, whereas the horizontal slashed plate is adjacent to the cells z_2 and z_3 which are in the non-facet-to-facet position.

Definition 1.3.2. For $\mathsf{X} \in \{\mathsf{V}, \mathsf{E}, \mathsf{P}, \mathsf{Z}\}$ the probability measure $\mathbb{Q}_{\mathsf{X}}$ on $\mathscr{T}$ given by

$$\mathbb{Q}_{\mathsf{X}}(A) := \frac{1}{\gamma_{\mathsf{X}}\lambda_3(B)}\mathbb{E}\sum_{\mathsf{x}\in\mathsf{X}} \mathbf{1}_B(c(\mathsf{x}))\mathbf{1}_A(\mathsf{T} - c(\mathsf{x}))$$

for $A \in \mathcal{T}$ and $B \in \mathcal{B}(\mathbb{R}^3)$ with $0 < \lambda_3(B) < \infty$ is called the *Palm distribution of the stationary random spatial tessellation* T *with respect to the typical* X*-type object*.

We emphasize that the probability measure $\mathbb{Q}_{\mathsf{X}}$ in Definition 1.3.2 is different from the probability measure $\mathbb{Q}_{\mathsf{X}}$ in Equations (4), (5) and (8).

Remark 1.3.3. Let $\mathsf{X} \in \{\mathsf{E}, \mathsf{P}, \mathsf{Z}\}$. From the fact that the function $m_{\mathsf{Y}}(\mathsf{x})$ does not only depend on the set X but also on the set Y, we cannot introduce the typical object of X by means of the grain distribution of the stationary simple particle process X in this case. The remark is still true when we consider other functions of X-type objects depending on the structure of the whole tessellation T.

Example 1.3.4. Consider an arbitrary stationary random spatial tessellation and its two primitive element sets $\mathsf{X} = \mathsf{E}$ and $\mathsf{Y} = \mathsf{V}$. Based on the fact that each edge $\mathsf{e} \in \mathsf{E}$ always has two end points which belong to V and does not have any internal vertices, we deduce that $m_{\mathsf{V}}(\mathsf{e}) = 2$ for all $\mathsf{e} \in \mathsf{E}$. It leads to $\mu_{\mathsf{EV}} = 2$. Vice versa, if E and V interchange the position then we cannot determine the mean value μ_{VE} without any further information about the given random tessellation.

Example 1.3.5. For a random spatial tessellation, $\mu_{\mathsf{PZ}} = 2$ because any plate of the tessellation is the intersection of two cells; see Figure 4.

Definition 1.3.6. Because of combinatorial and topological relations within any particular spatial tessellation the twelve adjacency mean values μ_{XY}, for $\mathsf{X}, \mathsf{Y} \in \{\mathsf{V}, \mathsf{E}, \mathsf{P}, \mathsf{Z}\}$ and $\mathsf{X} \neq \mathsf{Y}$, can be expressed for a random tessellation as functions of just three parameters; they have cyclic subscripts.

μ_{VE} – the mean number of edges emanating from the typical vertex,
μ_{EP} – the mean number of plates emanating from the typical edge and
μ_{PV} – the mean number of vertices on the boundary of the typical plate;

see [**38**] or also [**27**] and [**14**] where another notation is used. Note that all twelve adjacencies are invariant under topological transformations of $\mathbb{R}^3$, defined in [**7**, page 166]. So we call μ_{VE}, μ_{EP} and μ_{PV} the *topological parameters*.

Remark 1.3.7. In a facet-to-facet three-dimensional tessellation, the j-dimensional faces of an X-type object are primitive elements. In contrast, for non facet-to-facet cases we must carefully distinguish between the primitive elements and the j-faces of polytopes. For example a cell can have vertices on its boundary which are not 0-dimensional faces (apices) of that polytope. A 1-dimensional face (ridge) of a cell can have vertices in its relative interior, this being impossible for edges. A 2-dimensional face (facet) of a cell may not be a plate. Hence we use the notation X_j for the class of all j-dimensional faces of X-type objects, $j < \dim(\mathsf{X}\text{-type object})$. We emphasize that in the class X_j, the multiplicities of the elements are allowed. We pay attention to the case when $\mathsf{X} = \mathsf{Z}$ and $j = 0, 1, 2$ and obtain three corresponding classes

Z_0 – the class of apices of all cells,
Z_1 – the class of ridges of all cells,
Z_2 – the class of facets of all cells.

Another important class is P_1 - the class of the 1-dimensional faces of all plates, called the *plate-sides*.

Remark 1.3.8. We notice that in general, Z_0, Z_1, Z_2 and P_1 are *multisets* because of the multiplicities of the elements as we have discussed in Remark 1.3.7. These multisets will be defined later in Definition 2.2.6. For example if k cells have a 0-face located at a vertex $\mathsf{v} \in \mathsf{V}$, then the multiset Z_0 has k elements positioned on v; note that here $k \leq m_{\mathsf{Z}}(\mathsf{v})$. This example shows us the difference between the non-multiset V of vertices and the multiset Z_0 of cell-apices. Actually, we can understand Z_0 as a stationary point process in $\mathbb{R}^3$ which might be not simple. Similarly, Z_1, Z_2 and P_1 are (possibly not simple) stationary particle processes in $\mathbb{R}^3$.

Furthermore, we have

Definition 1.3.9. For $\mathsf{X} \in \{\mathsf{E}, \mathsf{P}, \mathsf{Z}\}$ and $j<\dim(\mathsf{X}\text{-type object})$, we define $n_j(\mathsf{x})$ as the number of j-faces of a particular object $\mathsf{x} \in \mathsf{X}$ and $\nu_j(\mathsf{X}) := \mathbb{E}_{\mathsf{X}}[n_j(\mathsf{x})]$ is the mean number of j-faces of the typical X-type object. For example it is

$\nu_0(\mathsf{Z})$ – the mean number of apices of the typical cell,
$\nu_1(\mathsf{Z})$ – the mean number of ridges of the typical cell,
$\nu_2(\mathsf{Z})$ – the mean number of facets of the typical cell.

Remark 1.3.10. $\mathbb{E}_{\mathsf{X}}$ in Definition 1.3.9 is the expectation with respect to the grain distribution $\mathbb{Q}_{\mathsf{X}}$ of the stationary simple particle process X as in (4).

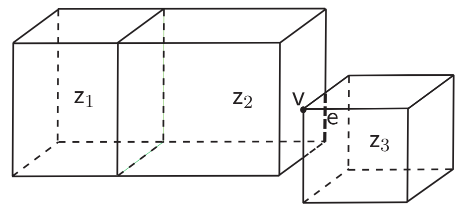

FIGURE 5. A π-edge e and a hemi-vertex v

Remark 1.3.11. Sometimes we use the notation $\mathsf{X}[\cdot]$ for a subset of the set X, where the term in brackets is a suitably chosen symbol describing the property of the subset. For example, the subsets of horizontal and vertical edges are denoted by $\mathsf{E}[\mathsf{hor}]$ and $\mathsf{E}[\mathsf{vert}]$ respectively.

Definition 1.3.12. We call an edge whose relative interior is contained in the relative interior of a cell-facet a *π-edge* and a vertex in the relative interior of a cell-facet a *hemi-vertex*, see [**38**].

An example of a π-edge and a hemi-vertex is illustrated in Figure 5. The subsets of π-edges and hemi-vertices are denoted by $\mathsf{E}[\pi]$ and $\mathsf{V}[\mathsf{hemi}]$ in that order.

Definition 1.3.13. If a spatial tessellation is not facet-to-facet, a face of a primitive element can have interior structure. To quantify the effects of this phenomenon four additional parameters for a random tessellation were introduced in [**38**]. They are called *interior parameters* and defined as follows:

- ξ – the proportion of π-edges in the tessellation: $\xi = \gamma_{\mathsf{E}[\pi]}/\gamma_{\mathsf{E}}$,
- κ – the proportion of hemi-vertices in the tessellation: $\kappa = \gamma_{\mathsf{V}[\mathsf{hemi}]}/\gamma_{\mathsf{V}}$,
- ψ – the mean number of relative ridge-interiors adjacent to the typical vertex: $\psi = \mu_{\mathsf{V}\mathring{\mathsf{Z}}_1}$,
- τ – the mean number of relative plate-side-interiors adjacent to the typical vertex: $\tau = \mu_{\mathsf{V}\mathring{\mathsf{P}}_1}$,

where $\mathring{\mathsf{X}}$ is the set of relative interiors of members of X.

Remark 1.3.14. Note that the interior parameters ξ and κ using the adjacency notation can be written as $\xi = \mu_{\mathring{\mathsf{E}}\mathring{\mathsf{Z}}_2}$ and $\kappa = \mu_{\mathsf{V}\mathring{\mathsf{Z}}_2}$. Indeed, by definition we have, for $B \in \mathcal{B}(\mathbb{R}^3)$ with $0 < \lambda_3(B) < \infty$,

$$\mu_{\mathring{\mathsf{E}}\mathring{\mathsf{Z}}_2} = \frac{1}{\gamma_{\mathsf{E}}\lambda_3(B)}\mathbb{E}\sum_{\mathsf{e}\in\mathsf{E}} \mathbf{1}_B(c(\mathsf{e}))m_{\mathring{\mathsf{Z}}_2 - c(\mathsf{e})}(\mathring{\mathsf{e}} - c(\mathsf{e})).$$

From the fact that each relative edge-interior is adjacent to either 0 or 1 relative cell-facet-interiors, we find that

$$\mu_{\overset{\circ}{\mathsf{E}}\overset{\circ}{\mathsf{Z}}_2} = \frac{1}{\gamma_{\mathsf{E}}\lambda_3(B)}\mathbb{E}\sum_{\mathsf{e}\in\mathsf{E}:m_{\overset{\circ}{\mathsf{Z}}_2}(\overset{\circ}{\mathsf{e}})=1}\mathbf{1}_B(c(\mathsf{e})) = \frac{1}{\gamma_{\mathsf{E}}}\cdot\frac{1}{\lambda_3(B)}\mathbb{E}\sum_{\mathsf{e}\in\mathsf{E}[\pi]}\mathbf{1}_B(c(\mathsf{e})) = \frac{\gamma_{\mathsf{E}[\pi]}}{\gamma_{\mathsf{E}}}.$$

The argument for κ is similar. Naturally all four interior parameters ξ, κ, ψ and τ are zero in the facet-to-facet case. In the proof of [**6**, Lemma 1] it is shown that a random spatial tessellation is facet-to-facet if and only if $\xi = 0$.

The initial template for the construction of a column tessellation (the main focus of Chapter 2) is a stationary random planar tessellation, denoted by $\mathcal{Y}'$. The sets of planar primitive elements of $\mathcal{Y}'$ are V' (vertices), E' (edges) and Z' (cells). Analogously to the spatial case these entities are defined for planar tessellations; see [**5**]. Note that, V' is a stationary simple point process in $\mathbb{R}^2$, whereas E' and Z' are stationary simple particle processes in $\mathbb{R}^2$. For $\mathsf{X}', \mathsf{Y}' \in \{\mathsf{V}', \mathsf{E}', \mathsf{Z}'\}$ and $\mathsf{X}' \neq \mathsf{Y}'$, the intensities and the adjacency mean values are $\gamma_{\mathsf{X}'}$ and $\mu_{\mathsf{X}'\mathsf{Y}'}$, respectively. Formally, if $m_{\mathsf{Y}'}(\mathsf{x}')$ is the number of Y'-type objects adjacent to an element $\mathsf{x}' \in \mathsf{X}'$ then $\mu_{\mathsf{X}'\mathsf{Y}'} := \mathbb{E}_{\mathsf{X}'}[m_{\mathsf{Y}'}(\mathsf{x}')]$. Similar to the 3-dimensional version, $\mathbb{E}_{\mathsf{X}'}$ denotes the expectation with respect to a probability measure $\mathbb{Q}_{\mathsf{X}'}$ defined as follows.

Definition 1.3.15. For $\mathsf{X}' \in \{\mathsf{V}', \mathsf{E}', \mathsf{Z}'\}$ the probability measure $\mathbb{Q}_{\mathsf{X}'}$ on $\mathscr{T}'$ given by

$$\mathbb{Q}_{\mathsf{X}'}(A) := \frac{1}{\gamma_{\mathsf{X}'}\lambda_2(B)}\mathbb{E}\sum_{\mathsf{x}'\in\mathsf{X}'}\mathbf{1}_B(c(\mathsf{x}'))\mathbf{1}_A(\mathcal{Y}' - c(\mathsf{x}'))$$

for $A \in \mathcal{T}'$ and $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$ is called the *Palm distribution of the stationary random planar tessellation $\mathcal{Y}'$ with respect to the typical X'-type object*.

For instance, it is easy to see that $\mu_{\mathsf{E}'\mathsf{Z}'} = 2$. Additionally, for $\mathsf{X}' \in \{\mathsf{E}', \mathsf{Z}'\}$ and $j <$ (dim X'-type object), denote by $n_j(\mathsf{x}')$ the number of j-dimensional faces of a particular object $\mathsf{x}' \in \mathsf{X}'$ and $\nu_j(\mathsf{X}') := \mathbb{E}_{\mathsf{X}'}[n_j(\mathsf{x}')]$ the mean number for the typical X'-type object. We emphasize that if we work with the function $n_j(\mathsf{x}')$, it is enough to understand $\mathbb{E}_{\mathsf{X}'}$ as the expectation with respect to the grain distribution of the stationary particle process X'. In this situation we do not need the Palm distribution $\mathbb{Q}_{\mathsf{X}'}$ of the whole tessellation $\mathcal{Y}'$ with respect to the typical X'-type object in Definition 1.3.15. For example

$\nu_0(\mathsf{Z}')$ – the mean number of 0-dimensional faces of the typical cell,
$\nu_1(\mathsf{Z}')$ – the mean number of 1-dimensional faces (sides) of the typical cell.

Denote by Z'_0 and Z'_1 the class of 0-faces and the class of sides of all cells of $\mathcal{Y}'$ in that order. Note that in general, Z'_0 and Z'_1 are multisets. These multisets will be defined later in Definition 2.1.4. If the stationary random planar tessellation $\mathcal{Y}'$ is not side-to-side then $\mathcal{Y}'$ has vertices located in the relative interior of cell-sides. We call them *π-vertices*, because one angle created by the emanating edges is equal to π. We denote a vertex v' which is a π-vertex by $\mathsf{v}'[\pi]$ and the subset of all π-vertices by $\mathsf{V}'[\pi]$. The subset of non-π-vertices is denoted by $\mathsf{V}'[\bar{\pi}]$. An example of

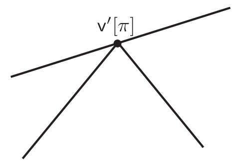

FIGURE 6. A π-vertex $\mathsf{v}'[\pi]$

a π-vertex is shown in Figure 6. The interior parameter of the stationary random planar tessellation $\mathcal{Y}'$ is

ϕ – the proportion of π-vertices in the tessellation: $\phi = \gamma_{\mathsf{V}'[\pi]}/\gamma_{\mathsf{V}'}$; formally $\phi = \mu_{\mathsf{V}'\mathring{\mathsf{Z}}'_1}$ (the argument is similar to Remark 1.3.14).

Remark 1.3.16. For stationary random 3-dimensional tessellations (stationary random planar tessellations, respectively), if X and Y (X' and Y', respectively) are both sets of primitive elements, it has been proved for example in [**13, 27, 14, 40, 18, 11**] that

$$\lambda_{\mathsf{X}}\mu_{\mathsf{XY}} = \lambda_{\mathsf{Y}}\mu_{\mathsf{YX}} \quad (\lambda_{\mathsf{X}'}\mu_{\mathsf{X}'\mathsf{Y}'} = \lambda_{\mathsf{Y}'}\mu_{\mathsf{Y}'\mathsf{X}'}, \text{ respectively}).$$

A more general approach to this identity is established in [**18**]. This extension remains true for sets of primitive elements and adjacency relation. Recently, some generalizations of this extension, namely, for sets of faces of primitive elements and for any arbitrary symmetric relation, are obtained in the work of Weiss and Cowan [**39**].

1.4. STIT tessellations

The notion of STIT tessellations is introduced by Nagel and Weiß in 2005 [**21**]. Results on this model in $\mathbb{R}^d$ are contained in Chapter 4. STIT tessellations are not face-to-face; recall Definition 1.2.8 for the face-to-face property of d-dimensional tessellations. In order to make a good connection between the construction of this model and its geometric description, we introduce firstly a construction of planar STIT tessellations in a bounded polygon before a formal construction in higher dimensions.

1.4.1. Construction of planar STIT tessellations in a bounded polygon. Within a polygon $W \subset \mathbb{R}^2$ with positive area, the construction is as follows. At first, W is equipped with a random lifetime. When the lifetime of W runs out, we choose a random line L, which divides W into two non-empty sub-polygons $W \cap \mathsf{L}^+$ and $W \cap \mathsf{L}^-$, where L^+ and L^- are two closed half-planes specified by L. Now, the (random) construction continues independently and recursively in $W \cap \mathsf{L}^+$ and $W \cap \mathsf{L}^-$ until some fixed time threshold $t > 0$ is reached; see Figure 7. The outcome $Y(t, W)$ of this algorithm is a random subdivision of W into polygons (cells). We call $(Y(t, W), t > 0)$ a *planar STIT tessellation process within* W.

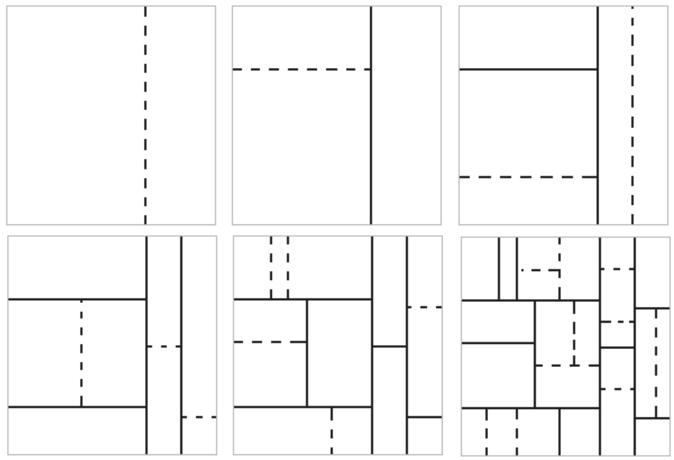

FIGURE 7. States of the random cell division process at different time instants; the respective new segments are dashed. Here, W is a square and $\mathbb{Q}$ is concentrated on two orthogonal directions.

In detail, we have to specify the lifetime distribution of the cells and the law of the cell-separating lines. For this purpose, recall that $A(2,1)$ is the set of lines in $\mathbb{R}^2$ and $G(2,1)$ is the subset of those lines going through the origin. Furthermore, let Λ be a locally finite, translation invariant measure on $A(2,1)$. According to [**30**, Theorem 4.4.1], the measure Λ admits the decomposition

$$\int\limits_{A(2,1)} f(L)\,\Lambda(\mathrm{d}L) = \gamma \int\limits_{G(2,1)} \int\limits_{L_0^\perp} f(L_0 + x)\,\lambda_{L_0^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L_0), \tag{9}$$

for any non-negative measurable function f on $A(2,1)$. Here $\lambda_{L_0^\perp}$ is the Lebesgue measure on the orthogonal complement $L_0^\perp$ of $L_0 \in G(2,1)$ and $\mathbb{Q}$ is a probability measure on $G(2,1)$. We require that $\mathbb{Q}$ is non-degenerate in the sense that $\mathbb{Q}$ does not concentrate on one direction. In other words, $|\operatorname{supp}\mathbb{Q}| \geq 2$, recalling that $|\cdot|$ stands for the cardinality. We assume $\gamma = 1$ in the decomposition (9). This means

$$\int\limits_{A(2,1)} f(L)\,\Lambda(\mathrm{d}L) = \int\limits_{G(2,1)} \int\limits_{L_0^\perp} f(L_0 + x)\,\lambda_{L_0^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L_0). \tag{10}$$

For a set $B \subset \mathbb{R}^2$, we write $\langle B \rangle = \{L \in A(2,1) : L \cap B \neq \emptyset\}$ for the set of lines intersecting B. Suppose that $\mathsf{z}' \subset W$ is a cell which is probably subdivided further. Let z' be a realization of the cell z'. Concentrating on the cell z', let $\beta(z')$ denote its birth-time and give an independent and identically distributed sequence $\{(\tilde{\tau}_j, \mathsf{L}_j)\}$, $j = 1, 2, \ldots$ of pairs of independent random variables. Here, $\tilde{\tau}_j$ is exponentially distributed with parameter $\Lambda(\langle W \rangle)$ and L_j is a random line with distribution $\frac{\Lambda(\cdot \cap \langle W \rangle)}{\Lambda(\langle W \rangle)}$, that is, for any $A \in \mathcal{B}(A(2,1))$,

$$\mathbb{P}(\mathsf{L}_j \in A) = \frac{\Lambda(A \cap \langle W \rangle)}{\Lambda(\langle W \rangle)}.$$

Consider each $\tilde{\tau}_j$ as the length of a time interval. Then for $i = 1, 2, \dots$ we can consider $\beta(z') + \sum_{j=1}^{i} \tilde{\tau}_j$ as the time when the random line L_i is thrown into W. We have to look whether L_i intersects z' or not. At the moment of division, the lifetime of the cell z' ends. Simultaneously, z' is divided into two new smaller cells whose independent lifetimes begin.

The next proposition gives us the conditional lifetime distribution of z'.

Proposition 1.4.1. *The conditional lifetime of the cell z' given a realization z' of z' is exponentially distributed with parameter $\Lambda(\langle z' \rangle)$.*

Proof. The conditional lifetime of the cell z' given a realization z' of z' is the lifetime of z' denoted by $\tau(z')$. Fix $s > 0$. Put $\mathsf{i}^* := \min\{j \in \mathbb{N} : \mathsf{L}_j \cap z' \neq \emptyset\}$. Then L_i is called the dividing-hyperplane of z' if $\mathsf{i}^* = i$. We have

$$\begin{aligned}
\mathbb{P}(\tau(z') > s) &= \sum_{i=1}^{\infty} \mathbb{P}(\tau(z') > s | \mathsf{i}^* = i) \mathbb{P}(\mathsf{i}^* = i) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(\sum_{j=1}^{i} \tilde{\tau}_j > s) \left[1 - \frac{\Lambda(\langle z' \rangle)}{\Lambda(\langle W \rangle)}\right]^{i-1} \frac{\Lambda(\langle z' \rangle)}{\Lambda(\langle W \rangle)}.
\end{aligned}$$

Due to the independence of the $\tilde{\tau}_j$, $j = 1, 2, \dots$ and the fact that each $\tilde{\tau}_j$ is exponentially distributed with parameter $\Lambda(\langle W \rangle)$, the distribution of the random time $\sum_{j=1}^{i} \tilde{\tau}_j$ is the Gamma distribution with parameter $(i, \Lambda(\langle W \rangle))$. Consequently,

$$\begin{aligned}
\mathbb{P}(\tau(z') > s) &= \sum_{i=1}^{\infty} \int_s^{\infty} \frac{x^{i-1} \Lambda(\langle W \rangle)^i e^{-\Lambda(\langle W \rangle) x}}{(i-1)!} \mathrm{d}x \left[1 - \frac{\Lambda(\langle z' \rangle)}{\Lambda(\langle W \rangle)}\right]^{i-1} \frac{\Lambda(\langle z' \rangle)}{\Lambda(\langle W \rangle)} \\
&= \int_s^{\infty} \sum_{i=1}^{\infty} \frac{x^{i-1} \Lambda(\langle W \rangle)^i e^{-\Lambda(\langle W \rangle) x}}{(i-1)!} \left[1 - \frac{\Lambda(\langle z' \rangle)}{\Lambda(\langle W \rangle)}\right]^{i-1} \frac{\Lambda(\langle z' \rangle)}{\Lambda(\langle W \rangle)} \mathrm{d}x \\
&= \Lambda(\langle z' \rangle) \int_s^{\infty} \sum_{i=1}^{\infty} \frac{x^{i-1} [\Lambda(\langle W \rangle) - \Lambda(\langle z' \rangle)]^{i-1}}{(i-1)!} e^{-\Lambda(\langle W \rangle) x} \mathrm{d}x \\
&= \Lambda(\langle z' \rangle) \int_s^{\infty} e^{[\Lambda(\langle W \rangle) - \Lambda(\langle z' \rangle)] x} e^{-\Lambda(\langle W \rangle) x} \mathrm{d}x = \Lambda(\langle z' \rangle) \int_s^{\infty} e^{-\Lambda(\langle z' \rangle) x} \mathrm{d}x = e^{-\Lambda(\langle z' \rangle) s}
\end{aligned}$$

which completes our proof. □

Proposition 1.4.2. *The conditional dividing-hyperplane distribution of the cell z' given a realization z' of z' is $\Lambda(\cdot \cap \langle z' \rangle)/\Lambda(\langle z' \rangle)$.*

Proof. Indeed, because the random lines L_j, $j = 1, 2, \dots$ are independent, for any $A \in \mathcal{B}(A(2,1))$, we have

$$\mathbb{P}(\mathsf{L}_i \in A | \mathsf{i}^* = i) = \frac{\mathbb{P}(\mathsf{L}_i \in A, \mathsf{i}^* = i)}{\mathbb{P}(\mathsf{i}^* = i)},$$

which is

$$\frac{\mathbb{P}(\mathsf{L}_i \in A \cap \langle z' \rangle, \mathsf{L}_1 \notin \langle z' \rangle, \mathsf{L}_2 \notin \langle z' \rangle, \dots, \mathsf{L}_{i-1} \notin \langle z' \rangle)}{\mathbb{P}(\mathsf{L}_i \in \langle z' \rangle, \mathsf{L}_1 \notin \langle z' \rangle, \mathsf{L}_2 \notin \langle z' \rangle, \dots, \mathsf{L}_{i-1} \notin \langle z' \rangle)}$$
$$= \frac{\mathbb{P}(\mathsf{L}_i \in A \cap \langle z' \rangle) \prod_{j=1}^{i-1} \mathbb{P}(\mathsf{L}_j \notin \langle z' \rangle)}{\mathbb{P}(\mathsf{L}_i \in \langle z' \rangle) \prod_{j=1}^{i-1} \mathbb{P}(\mathsf{L}_j \notin \langle z' \rangle)} = \frac{\frac{\Lambda(A \cap \langle z' \rangle \cap \langle W \rangle)}{\Lambda(\langle W \rangle)}}{\frac{\Lambda(\langle z' \rangle \cap \langle W \rangle)}{\Lambda(\langle W \rangle)}} = \frac{\Lambda(A \cap \langle z' \rangle)}{\Lambda(\langle z' \rangle)}.$$

□

Note that, if $\mathbb{Q}$ in (10) is rotation invariant on $G(2,1)$ then $\Lambda(\langle z' \rangle)$ is just the perimeter of the polygon z' up to a constant factor. This factor can be calculated explicitly.

Proposition 1.4.3. *If the locally finite translation invariant measure Λ on $A(2,1)$ is chosen as in (10) and the probability measure $\mathbb{Q}$ in this decomposition is rotation invariant on $G(2,1)$ then*

$$\Lambda(\langle z' \rangle) = \frac{1}{\pi} \ell(z'),$$

where $\ell(z')$ is the perimeter of the planar cell z'.

Proof. Indeed, if ν_1 denotes the unique rotation invariant probability measure on $G(2,1)$; see [**30**, Theorem 13.2.11], we infer that $\mathbb{Q}$ must be equal to ν_1. Denote by χ the Euler characteristic, it holds that

$$\Lambda(\langle z' \rangle) = \int\limits_{A(2,1)} \chi(z' \cap L) \Lambda(\mathrm{d}L) = \int\limits_{G(2,1)} \int\limits_{L_0^\perp} \chi(z' \cap (L_0 + x))\, \lambda_{L_0^\perp}(\mathrm{d}x)\, \nu_1(\mathrm{d}L_0)$$
$$= \int\limits_{G(2,1)} \lambda_{L_0^\perp}(z' | L_0^\perp) \nu_1(\mathrm{d}L_0),$$

where $z'|L_0^\perp$ denotes the image of z' under orthogonal projection to the subspace $L_0^\perp$. According to [**30**, Equation (5.8)], the last expression is proportional to the intrinsic volume $V_1(z')$ (see [**30**, Section 14.2] for definitions of intrinsic volumes) as follows:

$$\Lambda(\langle z' \rangle) = \int\limits_{G(2,1)} \lambda_{L_0^\perp}(z' | L_0^\perp) \nu_1(\mathrm{d}L_0) = \frac{\kappa_1^2}{2\kappa_2} V_1(z') = \frac{2}{\pi} V_1(z') = \frac{1}{\pi} \ell(z').$$

Here $\kappa_i := \lambda_i(B^i) = \pi^{\frac{i}{2}} / \Gamma\big(1 + \frac{i}{2}\big)$ is the volume of the i-dimensional unit ball. □

1.4.2. Formal construction of the d-dimensional STIT tessellations. Let $d \geq 2$. We start with a polytope of $\mathbb{R}^d$ with positive volume and a hyperplane measure which are also denoted by W and Λ, respectively. Now Λ is assumed to be a locally finite, translation invariant measure on the set $A(d, d-1)$ of all hyperplanes in $\mathbb{R}^d$. Recall that $G(d, d-1)$ denotes the subset of those hyperplanes containing the origin. Then according to [**30**, Theorem 4.4.1], the decomposition of Λ resembles the form of (9) in the sense that

$$\int\limits_{A(d,d-1)} f(H)\,\Lambda(\mathrm{d}H) = \gamma \int\limits_{G(d,d-1)} \int\limits_{H_0^\perp} f(H_0+x)\,\lambda_{H_0^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}H_0), \tag{11}$$

for any non-negative measurable function f on $A(d,d-1)$. Here $\lambda_{H_0^\perp}$ is the Lebesgue measure on the orthogonal complement $H_0^\perp$ of H_0 and $\mathbb{Q}$ now, without the danger of confusion, is a probability measure on $G(d,d-1)$. Furthermore, we assume that $\mathbb{Q}$ is non-degenerate, that is, $\operatorname{span}(\{H_0^\perp \cap \mathscr{S}_+^{d-1} : H_0 \in \operatorname{supp}\mathbb{Q}\}) = \mathbb{R}^d$, where $\mathscr{S}_+^{d-1}$ denotes the upper unit half-sphere in $\mathbb{R}^d$. From now on we assume $\gamma = 1$ in the decomposition (11). This means

$$\int\limits_{A(d,d-1)} f(H)\,\Lambda(\mathrm{d}H) = \int\limits_{G(d,d-1)} \int\limits_{H_0^\perp} f(H_0+x)\,\lambda_{H_0^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}H_0). \tag{12}$$

For a set $B \subset \mathbb{R}^d$, we write $\langle B\rangle = \{H \in A(d,d-1) : H \cap B \neq \emptyset\}$ for the set of hyperplanes intersecting B. Let τ_j, $j = 1,2,\ldots$, be a sequence of independent and identically distributed random variables, where τ_j is exponentially distributed with parameter 1. Our purpose is to construct a tessellation process $(Y(t,W), t \geq 0)$ within W, which is called a *d-dimensional STIT tessellation process within W*. We construct $Y(t,W)$ by induction.

- At time $t = 0$, we put $Y(0,W) := \{W\}$. The lifetime of W is given by

$$\tau(W) := \frac{1}{\Lambda(\langle W\rangle)}\tau_1.$$

  Because τ_1 is exponentially distributed with parameter 1, obviously, $\tau(W)$ is exponentially distributed with parameter $\Lambda(\langle W\rangle)$. Put $\mathsf{s}_1 := \tau(W)$ then s_1 is the division time of W and also the first jump time, which means that we still have $Y(t,W) = W$ for $0 \leq t < \mathsf{s}_1$. At the jump time s_1, W is divided by a random hyperplane H_1 with distribution $\Lambda(\cdot \cap \langle W\rangle)/\Lambda(\langle W\rangle)$. Put

$$\mathsf{z}_1 := W \cap \mathsf{H}_1^+ \text{ and } \mathsf{z}_2 := W \cap \mathsf{H}_1^-,$$

  where H_1^+ and H_1^- are two closed half-spaces specified by H_1. We call W the (unique) corresponding cell of H_1. Moreover, H_1 is called the dividing-hyperplane of W. A cell-splitting hyperplane piece, namely, $W \cap \mathsf{H}_1$, is generated. Thus,

$$Y(\mathsf{s}_1, W) = \{\mathsf{z}_1, \mathsf{z}_2\}.$$

  Let z_j be a realization of the cell z_j, $j = 1,2$. Moreover, let $\beta(z)$ and $\tau(z)$ denote the birth-time and the lifetime of some cell z. Obviously, $\beta(z_1) = \beta(z_2) = \mathsf{s}_1$. Therefore, the conditional birth-time of the cell z_j given a realization z_j of z_j is $\beta(z_j) = \mathsf{s}_1$, which is exponentially distributed with parameter $\Lambda(\langle W\rangle)$. The lifetimes of z_1 and z_2 are given by

$$\tau(z_1) := \frac{1}{\Lambda(\langle z_1\rangle)}\tau_2, \quad \tau(z_2) := \frac{1}{\Lambda(\langle z_2\rangle)}\tau_3.$$

Hence, the conditional lifetime of the cell z_j given a realization z_j of z_j is $\tau(z_j) = \frac{1}{\Lambda(\langle z_j \rangle)}\tau_{j+1}$, which is exponentially distributed with parameter $\Lambda(\langle z_j \rangle)$. The conditional division time of z_j given a realization z_j of z_j is $\beta(z_j) + \tau(z_j) = \mathsf{s}_1 + \frac{1}{\Lambda(\langle z_j \rangle)}\tau_{j+1}$. Here $j = 1, 2$.

- Induction step: Assume that the STIT tessellation $Y(s, W)$ has been constructed in the form $Y(s, W) = \{\mathsf{z}_1, \mathsf{z}_2, \ldots, \mathsf{z}_m\}$.
  Let $\{z_1, z_2, \ldots, z_m\}$ be a realization of $Y(s, W)$. For $j = 1, 2, \ldots, m$, each cell z_j given the realization z_j of z_j has the conditional birth-time $\beta(z_j)$, the conditional lifetime $\tau(z_j)$ which is exponentially distributed with parameter $\Lambda(\langle z_j \rangle)$ and the conditional division time $\beta(z_j) + \tau(z_j)$. Put
  $$\mathsf{s}_m := \min_{1 \le j \le m} \big(\beta(z_j) + \tau(z_j)\big)$$
  then s_m is the next jump time, i.e. $Y(t, W)$ still has the realization $\{z_1, z_2, \ldots, z_m\}$ for $s \le t < \mathsf{s}_m$. Assume that z_k is the cell satisfying $\beta(z_k) + \tau(z_k) = \mathsf{s}_m$. Then at the jump time s_m, z_k is divided by a random hyperplane H_m with distribution $\Lambda(\cdot \cap \langle z_k \rangle)/\Lambda(\langle z_k \rangle)$ into two new cells $z_k \cap \mathsf{H}_m^+$ and $z_k \cap \mathsf{H}_m^-$. The random hyperplane H_m is assumed to be independent of all random hyperplanes H_i, $i = 1, 2, \ldots, m-1$. We call z_k the (unique) corresponding cell of H_m. Moreover, H_m is called the dividing-hyperplane of z_k. A cell-splitting hyperplane piece, namely, $z_k \cap \mathsf{H}_m$, is generated. Thus $Y(\mathsf{s}_m, W)$ has the conditional realization
  $$\big(\{z_1, z_2, \ldots, z_m\} \setminus \{z_k\}\big) \cup \{z_k \cap H_m^+, z_k \cap H_m^-\}$$
  given a realization H_m of the random hyperplane H_m. Here the cell $z_k \cap \mathsf{H}_m^+$ given a realization H_m of H_m has the conditional birth-time $\beta(z_k \cap H_m^+) = \mathsf{s}_m$, the conditional lifetime $\tau(z_k \cap H_m^+)$ given by
  $$\tau(z_k \cap H_m^+) := \frac{1}{\Lambda(\langle z_k \cap H_m^+ \rangle)}\tau_{2m},$$
  which is exponentially distributed with parameter $\Lambda(\langle z_k \cap H_m^+ \rangle)$ and the conditional division time $\mathsf{s}_m + \frac{1}{\Lambda(\langle z_k \cap H_m^+ \rangle)}\tau_{2m}$. Whereas the cell $z_k \cap \mathsf{H}_m^-$ given a realization H_m of H_m has the conditional birth-time $\beta(z_k \cap H_m^-) = \mathsf{s}_m$, the conditional lifetime $\tau(z_k \cap H_m^-)$ given by
  $$\tau(z_k \cap H_m^-) := \frac{1}{\Lambda(\langle z_k \cap H_m^- \rangle)}\tau_{2m+1},$$
  which is exponentially distributed with parameter $\Lambda(\langle z_k \cap H_m^- \rangle)$ and the conditional division time $\mathsf{s}_m + \frac{1}{\Lambda(\langle z_k \cap H_m^- \rangle)}\tau_{2m+1}$.

Remark 1.4.4. The construction of the d-dimensional STIT tessellation $Y(t, W)$ in this subsection differs from the one (obviously generalized from the case of dimension 2) given in Subsection 1.4.1. The STIT tessellations constructed in these two ways have the same distribution because both constructions begin with the same hyperplane measure Λ and lead to the same conditional lifetime distribution as well as the conditional dividing-hyperplane distribution of an arbitrary cell z given a realization z of z. In particular, the conditional lifetime of z given a realization z of z is exponentially distributed with parameter $\Lambda(\langle z\rangle)$ and the conditional dividing-hyperplane distribution of z given a realization z of z is $\Lambda(\cdot \cap \langle z\rangle)/\Lambda(\langle z\rangle)$.

Besides of looking at the local tessellation $Y(t, W)$ within W, it is convenient to extend $Y(t, W)$ to a random tessellation $Y(t)$ in $\mathbb{R}^d$ in such a way that for any W as above, $Y(t)$ restricted to W has the same distribution as the previously constructed $Y(t, W)$ (this is possible by consistency according to [**21**, Theorem 1]). We call $Y(t)$ a *STIT tessellation* of $\mathbb{R}^d$ since $Y(t)$ enjoys a stochastic stability under iterations as explained later. It is easy to see that $Y(t)$ is stationary because Λ is translation invariant. For a global construction of $Y(t)$, a good reference is the paper of Mecke, Nagel and Weiß [**15**].

1.4.3. Associated objects of STIT tessellations. For $k = 0, \ldots, d$ we write $\mathscr{P}_k$ for the set of k-dimensional polytopes in $\mathbb{R}^d$ and $\mathscr{P}'_k := \mathscr{P}_k \setminus \{\emptyset\}$. With a d-dimensional STIT tessellation $Y(t)$, a number of geometric objects are associated. To introduce them, we write $\mathscr{M}\mathscr{P}_{d-1}^{(t)}$ for the set of cell-splitting hyperplane pieces of $Y(t)$, that is, until time t. According to the algorithm in Section 1.4.2, the random set of cell-splitting hyperplane pieces of the realization $\{z_1, z_2, \ldots, z_m\}$ of the local STIT tessellation $Y(t, W) = \{\mathsf{z}_1, \mathsf{z}_2, \ldots, \mathsf{z}_m\}$ is

$$\{\mathsf{H}_i \cap \text{ the corresponding cell of } \mathsf{H}_i : 1 \leq i \leq m-1\}.$$

When $d = 2$ these cell-splitting line pieces of $Y(t, W)$ are the dashed segments in Figure 7.

We emphasize that only the cell-splitting hyperplane pieces of the local STIT tessellation $Y(t, W)$ which do not intersect the boundary of the polytope W are members of $\mathscr{M}\mathscr{P}_{d-1}^{(t)}$. Each cell-splitting hyperplane piece of $Y(t, W)$ intersecting the boundary of W is only a subset of a cell-splitting hyperplane piece of the whole space STIT tessellation $Y(t)$ and consequently does not belong to the set $\mathscr{M}\mathscr{P}_{d-1}^{(t)}$.

More generally, for $k = 0, \ldots, d-2$ we denote by $\mathscr{M}\mathscr{P}_k^{(t)}$ the set of k-dimensional faces of members of $\mathscr{M}\mathscr{P}_{d-1}^{(t)}$. For $k = 0, \ldots, d-1$, it is also worthy to observe that $\mathscr{M}\mathscr{P}_k^{(t)}$ is a simple particle process in $\mathbb{R}^d$. In particular, $\mathscr{M}\mathscr{P}_k^{(t)}$ is a simple point process in $\mathscr{F}'$ – the system of non-empty closed subsets of $\mathbb{R}^d$ – which concentrates on $\mathscr{P}'_k$. We call $\mathscr{M}\mathscr{P}_k^{(t)}$ the process of *k-dimensional maximal polytopes* of $Y(t)$.

Example 1.4.5. $\mathscr{M}\mathscr{P}_1^{(t)}$ is the process of maximal segments. The notion of maximal segments was introduced by Mackisack and Miles [**12**]. A maximal segment (or an I-segment in some other literatures, for instance, [**33, 34, 16, 36**]) can be understood as a maximal union of collinear and connected line segments appearing in the union of all edges of a tessellation; see Figure 8 for an illustration of maximal segments of a planar STIT tessellation $Y(t)$.

The processes of k-dimensional maximal polytopes of $Y(t)$ for $k = 0, \ldots, d-1$ are the natural building blocks of $Y(t)$ and its lower-dimensional face-skeletons; see [**30**, Definition 10.1.4] for the notion of lower-dimensional face-skeletons of a

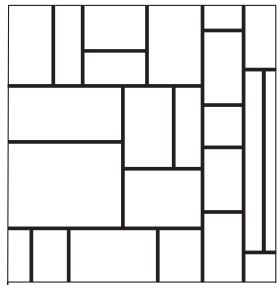

FIGURE 8. Here $d = 2$ and W is a square. The bold line segments which are maximal unions of some collinear bold line segments and do not intersect the boundary of W are maximal segments of the whole STIT tessellation $Y(t)$ in $\mathbb{R}^2$ and hence belong to $\mathscr{M}\mathscr{P}_1^{(t)}$. Each bold line segment which is a maximal union of some collinear bold line segments and which intersects the boundary of W is a maximal segment of the local STIT tessellation $Y(t, W)$ within W and is only a subset of a maximal segment of $Y(t)$ and hence does not belong to $\mathscr{M}\mathscr{P}_1^{(t)}$.

d-dimensional tessellation. We also consider k-dimensional weighted maximal polytopes, where the intrinsic volumes V_j, $0 \le j \le k$, constitute the weights. To define them, fix $k \in \{0, \dots, d-1\}$ and $j \in \{0, \dots, k\}$. Recall that $c(p)$ is the circumcenter of a polytope p. Moreover, write $\mathscr{P}_k^o$ for the measurable space of k-dimensional polytopes in $\mathbb{R}^d$ with circumcenter at the origin o (we equip $\mathscr{P}_k^o$ with the Hausdorff metric; see [**30**, Section 12.3] for the definition of the Hausdorff metric).

Definition 1.4.6. We introduce a probability measure $\mathbb{P}_{k,j}^{(t)}$ on $\mathscr{P}_k^o$ as follows:

$$\mathbb{P}_{k,j}^{(t)}(A) := \lim_{n\to\infty} \frac{\mathbb{E} \sum\limits_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\}\, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\mathbb{E} \sum\limits_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\}\, V_j(\mathsf{p})}, \tag{13}$$

where A is a Borel subset of $\mathscr{P}_k^o$ (following the proof of [**30**, Theorem 4.1.3(b)], it can be shown that the limit is well-defined). Another definition of $\mathbb{P}_{k,j}^{(t)}$ without using limit is

$$\mathbb{P}_{k,j}^{(t)}(A) := \frac{\mathbb{E} \sum\limits_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\mathbb{E} \sum\limits_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p})}, \tag{14}$$

where B is a Borel subset of $\mathbb{R}^d$ satisfying $0 < \lambda_d(B) < \infty$. A random polytope with distribution $\mathbb{P}_{k,j}^{(t)}$ is called a *V_j-weighted typical k-dimensional maximal polytope* of $Y(t)$ and will henceforth be denoted by $\mathsf{MP}_{k,j}^{(t)}$.

If $j = 0$, this is the typical k-dimensional maximal polytope and for $j = k$ we obtain the k-volume-weighted typical k-dimensional maximal polytope of the STIT tessellation $Y(t)$, which are two classical objects considered in stochastic geometry; see [**30, 29, 2**]. For example, $\mathsf{MP}_{1,0}^{(t)}$ is the *typical maximal segment*, whereas $\mathsf{MP}_{1,1}^{(t)}$ is the *length-weighted typical maximal segment*.

Remark 1.4.7. According to Theorem 1.1.11, two definitions of $\mathbb{P}_{k,j}^{(t)}$ are equivalent. Indeed, for $A \in \mathcal{B}(\mathscr{P}_k^o)$ and $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$ we rewrite the definitions (13) and (14) of $\mathbb{P}_{k,j}^{(t)}$ as follows:

$$\mathbb{P}_{k,j}^{(t)}(A) = \frac{\lim_{n\to\infty} \frac{1}{n^d}\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\}\, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\lim_{n\to\infty} \frac{1}{n^d}\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\}\, V_j(\mathsf{p})}, \tag{15}$$

$$\mathbb{P}_{k,j}^{(t)}(A) := \frac{\frac{1}{\lambda_d(B)}\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\frac{1}{\lambda_d(B)}\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p})}. \tag{16}$$

On the other hand, choose non-negative translation invariant measurable functions $\varphi_1, \varphi_2 : \mathscr{P}_k' \to \mathbb{R}$ given by $\varphi_1(p) := \mathbf{1}_A(p - c(p))\, V_j(p)$ and $\varphi_2(p) := V_j(p)$ for $p \in \mathscr{P}_k'$. Then both numerators in the right-hand sides of Equations (15) and (16) are the φ_1-density of the stationary simple particle process $\mathscr{M}\mathscr{P}_k^{(t)}$ in $\mathbb{R}^d$, whereas both denominators in the right-hand sides of these equations are the φ_2-density of $\mathscr{M}\mathscr{P}_k^{(t)}$.

Moreover, for $j = 0$, the probability measure $\mathbb{P}_{k,0}^{(t)}$ defined by

$$\begin{aligned}\mathbb{P}_{k,0}^{(t)}(A) &= \frac{\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)},\, c(\mathsf{p})\in B} \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))}{\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)},\, c(\mathsf{p})\in B} 1} \\ &= \lim_{n\to\infty} \frac{\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)},\, \mathsf{p}\subset[n]} \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))}{\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)},\, \mathsf{p}\subset[n]} 1}\end{aligned} \tag{17}$$

is the grain distribution of $\mathscr{M}\mathscr{P}_k^{(t)}$ because it has the forms of Equations (6) and (7).

A direct consequence of the definition of $\mathbb{P}_{k,j}^{(t)}$ is a relationship between $\mathsf{MP}_{k,j}^{(t)}$ and $\mathsf{MP}_{k,0}^{(t)}$.

Proposition 1.4.8. *Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$, $j \in \{0, \ldots, k\}$ and $f : \mathscr{P}_k^o \to \mathbb{R}$ be non-negative and measurable. A relationship between $\mathsf{MP}_{k,j}^{(t)}$ and $\mathsf{MP}_{k,0}^{(t)}$ is given by*

$$\mathbb{E} f(\mathsf{MP}_{k,j}^{(t)}) = [\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1} \mathbb{E}[f(\mathsf{MP}_{k,0}^{(t)}) V_j(\mathsf{MP}_{k,0}^{(t)})].$$

Proof. Recall that we use the notation $\gamma_{\mathscr{M}\mathscr{P}_k^{(t)}}$ for the intensity of the stationary simple particle process $\mathscr{M}\mathscr{P}_k^{(t)}$ in $\mathbb{R}^d$. For any Borel subset A of $\mathscr{P}_k^o$, applying Theorem 1.1.11(b) for $\mathscr{M}\mathscr{P}_k^{(t)}$ as well as two functions φ_1, φ_2 defined in Remark 1.4.7, we get

$$\mathbb{E}\mathbf{1}_A(\mathsf{MP}_{k,j}^{(t)}) = \mathbb{P}_{k,j}^{(t)}(A) = \frac{\lim\limits_{n\to\infty} \frac{1}{n^d} \mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\} \, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p})) \, V_j(\mathsf{p})}{\lim\limits_{n\to\infty} \frac{1}{n^d} \mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\} \, V_j(\mathsf{p})}$$

$$= \frac{\gamma_{\mathscr{M}\mathscr{P}_k^{(t)}} \int\limits_{\mathscr{P}_k^o} \mathbf{1}_A(p) \, V_j(p) \, \mathbb{P}_{k,0}^{(t)}(\mathrm{d}p)}{\gamma_{\mathscr{M}\mathscr{P}_k^{(t)}} \int\limits_{\mathscr{P}_k^o} V_j(p) \, \mathbb{P}_{k,0}^{(t)}(\mathrm{d}p)} = [\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1} \mathbb{E}[\mathbf{1}_A(\mathsf{MP}_{k,0}^{(t)}) V_j(\mathsf{MP}_{k,0}^{(t)})].$$

Thus the assertion holds for indicator functions of Borel subsets of $\mathscr{P}_k^o$, hence, also for linear combinations of such functions. By a standard argument of integration theory, it holds for any non-negative measurable function $f : \mathscr{P}_k^o \to \mathbb{R}$. That is

$$\mathbb{E} f(\mathsf{MP}_{k,j}^{(t)}) = [\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1} \mathbb{E}[f(\mathsf{MP}_{k,0}^{(t)}) V_j(\mathsf{MP}_{k,0}^{(t)})].$$

□

Corollary 1.4.9. *Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$, $i, j \in \{0, \ldots, k\}$ and $f : \mathscr{P}_k^o \to \mathbb{R}$ be non-negative and measurable. A relationship between $\mathsf{MP}_{k,i}^{(t)}$ and $\mathsf{MP}_{k,j}^{(t)}$ is given by*

$$\mathbb{E} f(\mathsf{MP}_{k,i}^{(t)}) = \frac{\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})}{\mathbb{E} V_i(\mathsf{MP}_{k,0}^{(t)})} \mathbb{E}[f(\mathsf{MP}_{k,j}^{(t)}) V_i(\mathsf{MP}_{k,j}^{(t)}) V_j(\mathsf{MP}_{k,j}^{(t)})^{-1}].$$

Proof. Using Proposition 1.4.8 with $f(\cdot)V_i(\cdot)V_j(\cdot)^{-1}$ instead of f there, we find that

$$\mathbb{E}[f(\mathsf{MP}_{k,j}^{(t)}) V_i(\mathsf{MP}_{k,j}^{(t)}) V_j(\mathsf{MP}_{k,j}^{(t)})^{-1}] = [\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1} \mathbb{E}[f(\mathsf{MP}_{k,0}^{(t)}) V_i(\mathsf{MP}_{k,0}^{(t)})].$$

On the other hand, using Proposition 1.4.8 again, the relationship between the V_i-weighted typical k-dimensional maximal polytope $\mathsf{MP}_{k,i}^{(t)}$ and the typical k-dimensional maximal polytope $\mathsf{MP}_{k,0}^{(t)}$ is given by

$$\mathbb{E} f(\mathsf{MP}_{k,i}^{(t)}) = [\mathbb{E} V_i(\mathsf{MP}_{k,0}^{(t)})]^{-1} \mathbb{E}[f(\mathsf{MP}_{k,0}^{(t)}) V_i(\mathsf{MP}_{k,0}^{(t)})].$$

We get

$$\mathbb{E} f(\mathsf{MP}_{k,i}^{(t)}) = \frac{\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})}{\mathbb{E} V_i(\mathsf{MP}_{k,0}^{(t)})} \mathbb{E}[f(\mathsf{MP}_{k,j}^{(t)}) V_i(\mathsf{MP}_{k,j}^{(t)}) V_j(\mathsf{MP}_{k,j}^{(t)})^{-1}].$$

□

We notice that any k-dimensional maximal polytope p of $Y(t)$ is the intersection of $(d-k)$ maximal polytopes of dimension $(d-1)$. In view of the spatio-temporal construction described in Subsection 1.4.2, each of these $(d-1)$-dimensional polytopes has a well-defined random birth-time. We denote the birth-times of these $(d-k)$ maximal polytopes by $\beta_1(\mathsf{p}), \ldots, \beta_{d-k}(\mathsf{p})$ and order them in such a way that $0 < \beta_1(\mathsf{p}) < \ldots < \beta_{d-k}(\mathsf{p}) < t$ holds almost surely.

Definition 1.4.10. For each k-dimensional maximal polytope $\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}$ we mark $c(\mathsf{p})$ with $\mathsf{p}_o := \mathsf{p} - c(\mathsf{p})$ and the last birth-time of p_o, namely, $\beta_{d-k}(\mathsf{p}_o) = \beta_{d-k}(\mathsf{p})$. This gives rise to a marked point process $\widehat{\mathscr{M}\mathscr{P}}_k^{(t)}$ in $\mathbb{R}^d \times \mathscr{P}_k^o \times (0,t)$. Now we introduce a probability measure $\widehat{\mathbb{P}}_{k,j}^{(t)}$ on $\mathscr{P}_k^o \times (0,t)$ as follows:

$$\widehat{\mathbb{P}}_{k,j}^{(t)}(A \times B_{d-k}) := \frac{\mathbb{E} \sum\limits_{(c(\mathsf{p}),\mathsf{p}_o,\beta_{d-k}(\mathsf{p}_o)) \in \widehat{\mathscr{M}\mathscr{P}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, \mathbf{1}_A(\mathsf{p}_o)\, V_j(\mathsf{p}_o)\, \mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{p}_o))}{\mathbb{E} \sum\limits_{(c(\mathsf{p}),\mathsf{p}_o,\beta_{d-k}(\mathsf{p}_o)) \in \widehat{\mathscr{M}\mathscr{P}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p}_o)}$$

where A is a Borel subset of $\mathscr{P}_k^o$, B is a Borel subset of $\mathbb{R}^d$ with $0 < \lambda_d(B) < \infty$ and B_{d-k} is a Borel subset of $(0,t)$. A pair of a random polytope and a random time with distribution $\widehat{\mathbb{P}}_{k,j}^{(t)}$ is called a *last-birth-time marked V_j-weighted typical k-dimensional maximal polytope* of $Y(t)$ and will be henceforth denoted by $(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$.

Remark 1.4.11. Note that for $j=0$, the probability measure $\widehat{\mathbb{P}}_{k,0}^{(t)}$ defined by

$$\widehat{\mathbb{P}}_{k,0}^{(t)}(A \times B_{d-k}) = \frac{\mathbb{E} \sum\limits_{(c(\mathsf{p}),\mathsf{p}_o,\beta_{d-k}(\mathsf{p}_o)) \in \widehat{\mathscr{M}\mathscr{P}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, \mathbf{1}_A(\mathsf{p}_o)\, \mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{p}_o))}{\mathbb{E} \sum\limits_{(c(\mathsf{p}),\mathsf{p}_o,\beta_{d-k}(\mathsf{p}_o)) \in \widehat{\mathscr{M}\mathscr{P}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))}$$

is the mark distribution of $\widehat{\mathscr{M}\mathscr{P}}_k^{(t)}$ according to Equation (8). In particular, $\widehat{\mathbb{P}}_{k,0}^{(t)}$ is the joint distribution of the two marks of the typical k-dimensional maximal polytope-circumcenter of $Y(t)$.

Proposition 1.4.12. *Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$, $j \in \{0, \ldots, k\}$ and $f : \mathscr{P}_k^o \times (0,t) \to \mathbb{R}$ be a non-negative and measurable function. A relationship between $(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$ and $(\mathsf{MP}_{k,0}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,0}^{(t)}))$ is given by*

$$\mathbb{E} f(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)})) = [\mathbb{E} V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1} \mathbb{E}[f(\mathsf{MP}_{k,0}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,0}^{(t)})) V_j(\mathsf{MP}_{k,0}^{(t)})].$$

Proof. It is easy to see that, for $A \in \mathcal{B}(\mathscr{P}_k^o)$ and $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$,

$$\widehat{\mathbb{P}}_{k,0}^{(t)}(A \times (0,t)) = \frac{\mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)},\, c(\mathsf{p}) \in B} \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))}{\mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)},\, c(\mathsf{p}) \in B} 1} = \mathbb{P}_{k,0}^{(t)}(A)$$

according to Equation (17). Applying Theorem 1.1.15 for the marked point process $\widehat{\mathscr{M}\mathscr{P}}_k^{(t)}$ in $\mathbb{R}^d$ with mark space $\mathscr{P}_k^o \times (0,t)$, we obtain, for $B_{d-k} \in \mathcal{B}((0,t))$,

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_A(\mathsf{MP}_{k,j}^{(t)})\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))] = \widehat{\mathbb{P}}_{k,j}^{(t)}(A \times B_{d-k}) \\
&= \frac{\gamma_{\widehat{\mathscr{M}\mathscr{P}}_k^{(t)}} \lambda_d(B) \int\limits_{\mathscr{P}_k^o \times (0,t)} \mathbf{1}_A(p)\mathbf{1}_{B_{d-k}}(\beta_{d-k})V_j(p)\widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,\beta_{d-k}))}{\gamma_{\widehat{\mathscr{M}\mathscr{P}}_k^{(t)}} \lambda_d(B) \int\limits_{\mathscr{P}_k^o \times (0,t)} V_j(p)\, \widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,\beta_{d-k}))} \\
&= \frac{\int\limits_{\mathscr{P}_k^o \times (0,t)} \mathbf{1}_A(p)\mathbf{1}_{B_{d-k}}(\beta_{d-k})V_j(p)\widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,\beta_{d-k}))}{\int\limits_{\mathscr{P}_k^o} V_j(p)\, \mathbb{P}_{k,0}^{(t)}(\mathrm{d}p)} \\
&= [\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1}\mathbb{E}[\mathbf{1}_A(\mathsf{MP}_{k,0}^{(t)})\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,0}^{(t)}))V_j(\mathsf{MP}_{k,0}^{(t)})].
\end{aligned}$$

By a standard argument of integration theory, we get the desired statement. □

Corollary 1.4.13. *Let $d \geq 2$, $k \in \{0,\dots,d-1\}$, $i,j \in \{0,\dots,k\}$ and $f : \mathscr{P}_k^o \times (0,t) \to \mathbb{R}$ be a non-negative and measurable function. A relationship between $(\mathsf{MP}_{k,i}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,i}^{(t)}))$ and $(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$ is given by*

$$\begin{aligned}
&\mathbb{E}f(\mathsf{MP}_{k,i}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,i}^{(t)})) \\
&\qquad = \frac{\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})}{\mathbb{E}V_i(\mathsf{MP}_{k,0}^{(t)})}\mathbb{E}[f(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))V_i(\mathsf{MP}_{k,j}^{(t)})V_j(\mathsf{MP}_{k,j}^{(t)})^{-1}].
\end{aligned}$$

Proof. Use Proposition 1.4.12. The method is similar to Corollary 1.4.9. □

Put $\Delta(t) := \{(r_1,\dots,r_{d-k}) \in \mathbb{R}^{d-k} : 0 < r_1 < \ldots < r_{d-k} < t\}$. Then $\Delta(t)$ is a $(d-k)$-simplex which is a subset of $\mathbb{R}^{d-k}$.

Definition 1.4.14. For each k-dimensional maximal polytope $\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}$ we mark $c(\mathsf{p})$ with $\mathsf{p}_o := \mathsf{p} - c(\mathsf{p})$ and the vector of $(d-k)$ birth-times of p_o, namely, $(\beta_1(\mathsf{p}_o),\dots,\beta_{d-k}(\mathsf{p}_o)) = (\beta_1(\mathsf{p}),\dots,\beta_{d-k}(\mathsf{p}))$. This gives rise to a marked point process $\widetilde{\mathscr{M}\mathscr{P}}_k^{(t)}$ in $\mathbb{R}^d \times \mathscr{P}_k^o \times \Delta(t)$. Now we introduce a probability measure $\widetilde{\mathbb{P}}_{k,j}^{(t)}$ on $\mathscr{P}_k^o \times \Delta(t)$ as follows:

$$\begin{aligned}
&\widetilde{\mathbb{P}}_{k,j}^{(t)}[A \times ((B_1 \times \ldots \times B_{d-k}) \cap \Delta(t))] = \left[\mathbb{E}\sum_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p})\right]^{-1} \times \\
&\times \mathbb{E} \sum_{(c(\mathsf{p}),\mathsf{p}_o,\beta_1(\mathsf{p}_o),\dots,\beta_{d-k}(\mathsf{p}_o)) \in \widetilde{\mathscr{M}\mathscr{P}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\mathbf{1}_A(\mathsf{p}_o)V_j(\mathsf{p}_o) \times \\
&\qquad \times \mathbf{1}\{0 < \beta_1(\mathsf{p}_o) < \ldots < \beta_{d-k}(\mathsf{p}_o) < t\}\mathbf{1}_{B_1}(\beta_1(\mathsf{p}_o)) \ldots \mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{p}_o))
\end{aligned}$$

where A is a Borel subset of $\mathscr{P}_k^o$, B is a Borel subset of $\mathbb{R}^d$ with $0 < \lambda_d(B) < \infty$ and $B_1,\dots,B_{d-k}$ are Borel subsets of $(0,t)$. A vector of a random polytope and $(d-k)$ random times with distribution $\widetilde{\mathbb{P}}_{k,j}^{(t)}$ is called a *birth-time-vector marked V_j-weighted* *typical k-dimensional maximal polytope* of $Y(t)$ and will be henceforth denoted by $(\mathsf{MP}_{k,j}^{(t)}, \beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$.

Proposition 1.4.15. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$, $i, j \in \{0, \dots, k\}$ and $f : \mathscr{P}_k^o \times (0,t)^{d-k} \to \mathbb{R}$ be non-negative and measurable. A relationship between $(\mathsf{MP}_{k,i}^{(t)}, \beta_1(\mathsf{MP}_{k,i}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,i}^{(t)}))$ and $(\mathsf{MP}_{k,j}^{(t)}, \beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$ is given by*

$$\begin{aligned}
&\mathbb{E}[f(\mathsf{MP}_{k,i}^{(t)}, \beta_1(\mathsf{MP}_{k,i}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,i}^{(t)}))\mathbf{1}_{\Delta(t)}((\beta_1(\mathsf{MP}_{k,i}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,i}^{(t)})))] \\
&= \frac{\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})}{\mathbb{E}V_i(\mathsf{MP}_{k,0}^{(t)})} \times \mathbb{E}[f(\mathsf{MP}_{k,j}^{(t)}, \beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\times \\
&\qquad \times \mathbf{1}_{\Delta(t)}((\beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)})))V_i(\mathsf{MP}_{k,j}^{(t)})V_j(\mathsf{MP}_{k,j}^{(t)})^{-1}].
\end{aligned}$$

Proof. Use the method in Proposition 1.4.12 and Corollary 1.4.9. □

1.4.4. Some properties of STIT tessellations. Fix $d \geq 2$. We collect in the following some important properties of STIT tessellations in $\mathbb{R}^d$; see [**15, 16, 36**].

[**Iteration of tessellations**]. Below we will exploit the fact that the tessellations $Y(t)$ are stable under iterations. To explain what this means, let $0 < s < t$ and define the *iteration* $Y(s) \boxplus Y(t)$ of $Y(s)$ and $Y(t)$ as the tessellation that arises by locally superimposing within the cells of $Y(s)$ independent copies of $Y(t)$. Formally, let $\{Y_i(t) : i \geq 1\}$ be a family of i.i.d. copies of $Y(t)$, which is also independent of $Y(s)$. Furthermore, let $\{\mathsf{z}_i : i \geq 1\}$ be an enumeration of the cells of $Y(s)$. Then,

$$Y(s) \boxplus Y(t) := \left\{\mathsf{z}_i \cap \mathsf{z} : \mathsf{z}_i \in Y(s),\ \mathsf{z} \in Y_i(t),\, \mathrm{int}\, \mathsf{z}_i \cap \mathrm{int}\, \mathsf{z} \neq \emptyset\right\}.$$

The STIT tessellations satisfy

$$Y(s) \boxplus Y(t) \overset{\mathcal{D}}{=} Y(s+t) \qquad \text{for any } s, t > 0. \tag{18}$$

In other words, the results are the same in distribution when we either run the cell-division algorithm in Subsection 1.4.2 from time s to time $s+t$ or perform at time s an iteration of $Y(s)$ and $Y(t)$. Moreover, they are stable under iterations in that the distributional equality

$$Y(t) \overset{\mathcal{D}}{=} n\,\big(\underbrace{Y(t) \boxplus \ldots \boxplus Y(t)}_{n \text{ times}}\big) \overset{\mathcal{D}}{=} nY(nt). \tag{19}$$

holds for any $n \in \mathbb{N}$; see [**21, 15, 16**]. Later this will play an important role in various proofs.

Remark 1.4.16. Equation (19) can be written without brackets between the STIT tessellations $Y(t)$ because for the tessellation process $(Y(t), t \geq 0)$, the operation $\boxplus$ is associative with respect to distribution. Indeed, using Equation (18) we find that

$$\begin{aligned}
(Y(t_1) \boxplus Y(t_2)) \boxplus Y(t_3) &\overset{\mathcal{D}}{=} Y(t_1+t_2) \boxplus Y(t_3) \overset{\mathcal{D}}{=} Y((t_1+t_2)+t_3) = \\
= Y(t_1+(t_2+t_3)) &\overset{\mathcal{D}}{=} Y(t_1) \boxplus Y(t_2+t_3) \overset{\mathcal{D}}{=} Y(t_1) \boxplus (Y(t_2) \boxplus Y(t_3))
\end{aligned}$$

for any $t_1, t_2, t_3 > 0$. It leads to

$$(Y(t_1) \boxplus Y(t_2)) \boxplus Y(t_3) \stackrel{\mathcal{D}}{=} Y(t_1) \boxplus Y(t_2) \boxplus Y(t_3).$$

In general, if T_1, T_2 and T_3 are arbitrary random tessellations then the distribution of $(\mathsf{T}_1 \boxplus \mathsf{T}_2) \boxplus \mathsf{T}_3$ must not be the same as the distribution of $\mathsf{T}_1 \boxplus (\mathsf{T}_2 \boxplus \mathsf{T}_3)$.

[**STIT scaling**]. We collect here two implications of the scaling property of a STIT tessellation $Y(t)$. Globally, it says that the dilation $tY(t)$ of $Y(t)$ by factor t has the same distribution as $Y(1)$, the STIT tessellation with time parameter 1, i.e.,

$$tY(t) \stackrel{\mathcal{D}}{=} Y(1) \quad \text{for all} \quad t > 0. \tag{20}$$

For a polytope W we also have the local scaling $tY(t, W) \stackrel{\mathcal{D}}{=} Y(1, tW)$; see [**21**] for example.

For $k \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, k\}$ let us denote by $\varrho_{k,j}^{(t)}$ the density of the jth intrinsic volume of $\mathscr{M}\mathscr{P}_k^{(t)}$, that is,

$$\begin{aligned} \varrho_{k,j}^{(t)} &:= \lim_{n\to\infty} \frac{1}{n^d} \mathbb{E} \sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\, \mathsf{p} \subset [n]\,\}\, V_j(\mathsf{p}) \\ &= \frac{1}{\lambda_d(B)} \mathbb{E} \sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p}) \end{aligned} \tag{21}$$

where $B \in \mathcal{B}$ satisfying $0 < \lambda_d(B) < \infty$. It is worthy to observe that $\varrho_{k,0}^{(t)} = \gamma_{\mathscr{M}\mathscr{P}_k^{(t)}} = \gamma_{\widehat{\mathscr{M}\mathscr{P}_k^{(t)}}} = \gamma_{\widetilde{\mathscr{M}\mathscr{P}_k^{(t)}}}$.

Using (20), the definitions (13) of $\mathbb{P}_{k,j}^{(t)}$ and (21) of $\varrho_{k,j}^{(t)}$ as well as the homogeneity of the intrinsic volumes in this order, one shows the following two facts.

Lemma 1.4.17. *For $t > 0$, $k \in \{0, \ldots, d-1\}$ and $i, j \in \{0, \ldots, k\}$ it holds that*

a) $\varrho_{k,j}^{(t)} = t^{d-j} \varrho_{k,j}^{(1)}$,

b) $\mathbb{E}V_i(\mathsf{MP}_{k,j}^{(t)}) = t^{-i}\, \mathbb{E}V_i(\mathsf{MP}_{k,j}^{(1)})$.

Proof. Note that $\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)} \Leftrightarrow t\mathsf{p} \in t\mathscr{M}\mathscr{P}_k^{(t)} \stackrel{\mathcal{D}}{=} \mathscr{M}\mathscr{P}_k^{(1)}$.
For (a):

$$\begin{aligned} \varrho_{k,j}^{(t)} &= \lim_{n\to\infty} \frac{1}{n^d} \mathbb{E} \sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\} V_j(\mathsf{p}) \\ &= \lim_{n\to\infty} \frac{1}{n^d} \mathbb{E} \sum_{t\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{t\mathsf{p} \subset [tn]\} V_j(\mathsf{p}) \\ &= \lim_{n\to\infty} \frac{1}{n^d} \mathbb{E} \sum_{\mathsf{p}'\in\mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\mathsf{p}' \subset [tn]\} V_j(\frac{1}{t}\mathsf{p}') \end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{t^d}{(tn)^d} \mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\mathsf{p}' \subset [tn]\} t^{-j} V_j(\mathsf{p}') \\
&= t^{d-j} \lim_{tn\to\infty} \frac{1}{(tn)^d} \mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\, \mathsf{p}' \subset [tn]\,\} V_j(\mathsf{p}') \\
&= t^{d-j} \lim_{n'\to\infty} \frac{1}{(n')^d} \mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\, \mathsf{p}' \subset [n']\,\} V_j(\mathsf{p}') = t^{d-j} \varrho_{k,j}^{(1)}.
\end{aligned}$$

For (b): The definition of $\mathbb{P}_{k,j}^{(t)}$ gives us

$$\begin{aligned}
\mathbb{E} V_i(\mathsf{MP}_{k,j}^{(t)}) &= \lim_{n\to\infty} \frac{\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\}\, V_i(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\mathbb{E} \sum_{\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}\{\mathsf{p} \subset [n]\}\, V_j(\mathsf{p})} \\
&= \lim_{n\to\infty} \frac{\mathbb{E} \sum_{t\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{t\mathsf{p} \subset [tn]\}\, V_i(\mathsf{p})\, V_j(\mathsf{p})}{\mathbb{E} \sum_{t\mathsf{p}\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{t\mathsf{p} \subset [tn]\}\, V_j(\mathsf{p})} \\
&= \lim_{n\to\infty} \frac{\mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\mathsf{p}' \subset [tn]\}\, V_i(\frac{1}{t}\mathsf{p}')\, V_j(\frac{1}{t}\mathsf{p}')}{\mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\mathsf{p}' \subset [tn]\}\, V_j(\frac{1}{t}\mathsf{p}')} \\
&= \lim_{n'\to\infty} \frac{t^{-i-j}\,\mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\mathsf{p}' \subset [n']\}\, V_i(\mathsf{p}')\, V_j(\mathsf{p}')}{t^{-j}\,\mathbb{E} \sum_{\mathsf{p}'\in \mathscr{M}\mathscr{P}_k^{(1)}} \mathbf{1}\{\mathsf{p}' \subset [n']\}\, V_j(\mathsf{p}')} \\
&= t^{-i}\, \mathbb{E} V_i(\mathsf{MP}_{k,j}^{(1)}).
\end{aligned}$$

□

[Poisson typical cell] Recall Example 1.1.5 for the definition of Poisson hyperplane processes and Example 1.2.4 for that of Poisson hyperplane tessellations. An important property of STIT tessellations is that: The distribution of the interior of the typical cell of the STIT tessellation $Y(t)$ is equal to the distribution of the interior of the typical cell of the stationary Poisson hyperplane tessellation $\mathrm{PHT}(t\Lambda)$ generated by the stationary Poisson hyperplane process $\mathrm{PHP}(t\Lambda)$ with intensity measure $t\Lambda$. Here the locally finite translation invariant measure Λ on $A(d, d-1)$ is chosen as in (12).

[Linear sections] The intersection of $Y(t)$ with a line L parallel to span u where $u \in \mathscr{S}_+^{d-1}$ is a stationary Poisson process in L with intensity $\Lambda(\langle [0, u] \rangle)t$.

CHAPTER 2

Column tessellations

An important problem in stochastic geometry is the development of mathematical models for random structures in materials science, geology, biology and other sciences whose properties are mathematically executable and not only tractable by simulation. Column tessellations are a new model of non-facet-to-facet random tessellations in $\mathbb{R}^3$, which could be contribute a solution to the above problem, at least to address crack structures in geology.

2.1. Construction

Based on the stationary random planar tessellation $\mathcal{Y}'$ in the horizontal plane $\mathcal{E} = \mathbb{R}^2 \times \{0\}$ we construct the spatial column tessellation $\mathcal{Y}$ in the following way:

For each cell z' of $\mathcal{Y}'$, we consider an infinite cylindrical column based on this cell and perpendicular to $\mathcal{E}$. Further we mark the circumcenter $c(\mathsf{z}')$ of z' with a random positive number $\rho_{\mathsf{z}'}$. Here $\rho : \mathscr{P}_2 \to (0, \infty)$ given by $\rho_{z'} := \rho(z')$ is a non-negative translation invariant function of z', for example, the area of this polygonal cell. Moreover, the set $\mathscr{P}_2$ is understood in the context $d = 2$: it is the set of 2-dimensional polytopes in $\mathbb{R}^2$. Such a mark ρ is created for all cells in $\mathcal{Y}'$. Now, for each planar cell z', we construct on the line going through $c(\mathsf{z}')$ perpendicular to $\mathcal{E}$ — we call this the *cell-axis* — a stationary and simple point process with intensity $\rho_{\mathsf{z}'}$. For all cells in $\mathcal{Y}'$, these cell-axis point processes are assumed to be conditionally independent given a realization of $\mathcal{Y}'$. In detail, let Φ_j, $j = 1, 2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) stationary simple point processes on $\{0\}^2 \times \mathbb{R}$ with intensity 1 and independent of $\mathcal{Y}' = \{\mathsf{z}'_j : j = 1, 2, \ldots\}$. Put $\mathsf{z}'_{jo} := \mathsf{z}'_j - c(\mathsf{z}'_j)$. Combining with the translation invariant property of the function ρ we choose $\frac{1}{\rho_{\mathsf{z}'_{jo}}}\Phi_j + c(\mathsf{z}'_j)$ as the point process on the cell-axis going through $c(\mathsf{z}'_j)$. Moreover, $\frac{1}{\rho_{\mathsf{z}'_{jo}}}\Phi_j$ is called the corresponding stationary simple point process of z'_j. To create the spatial tessellation, the column based on z' is intersected by horizontal cross-sections located at each of the random points of that column's point process. The resulting random three-dimensional tessellation $\mathcal{Y}$ is called a *column tessellation.* Note that the cell-axes do not belong to the column tessellation. Because the horizontal cross-sections are actually *plates* of $\mathcal{Y}$, and there are no other plates in the tessellation that are horizontal, we shall often refer to them as the *horizontal plates.* Any cell of $\mathcal{Y}$ is a right prism, where its base facet is a vertical translation of a cell of $\mathcal{Y}'$. Moreover we have the following property.

Property 2.1.1. *The simple superposition property:* For any vertex v' of the planar tessellation $\mathcal{Y}'$, the superposition of the corresponding stationary simple point processes of the adjacent cells of v' is a stationary simple point process on $\{0\}^2 \times \mathbb{R}$.

Proof. We can identify $\{0\}^2 \times \mathbb{R}$ with $\mathbb{R}$ using the bijection $f : \mathbb{R} \to \{0\}^2 \times \mathbb{R}$ given by $f(x) = (0, 0, x)$ for $x \in \mathbb{R}$. Let $\mathsf{M} := m_{\mathsf{Z}'}(\mathsf{v}')$ – the random number of adjacent cells of v' – and $\frac{1}{\rho_{\mathsf{z}'_{(1o)}}}\Phi_{(1)}, \ldots, \frac{1}{\rho_{\mathsf{z}'_{(\mathsf{M}o)}}}\Phi_{(\mathsf{M})}$ are the corresponding stationary simple point processes of adjacent cells $\mathsf{z}'_{(1)}, \ldots, \mathsf{z}'_{(\mathsf{M})}$ of v'. Recall that $\mathsf{z}'_{(jo)} = \mathsf{z}'_{(j)} - c(\mathsf{z}'_{(j)})$ for $j = 1, 2, \ldots, \mathsf{M}$. Denote by $\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}$ the superposition of $\frac{1}{\rho_{\mathsf{z}'_{(1o)}}}\Phi_{(1)}, \ldots, \frac{1}{\rho_{\mathsf{z}'_{(\mathsf{M}o)}}}\Phi_{(\mathsf{M})}$. From the fact that v' has at least 3 adjacent cells, we get

$$\begin{aligned}
&\mathbb{P}\Bigg(\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \text{ is a simple point process on } \mathbb{R}\Bigg)\\
&= \sum_{m=3}^{\infty} \mathbb{P}\Bigg(\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \text{ is a simple point process on } \mathbb{R}\Bigg|\mathsf{M} = m\Bigg)\mathbb{P}(\mathsf{M} = m)\\
&= \sum_{m=3}^{\infty} \mathbb{P}\Bigg(\bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \text{ is a simple point process on } \mathbb{R}\Bigg|\mathsf{M} = m\Bigg)\mathbb{P}(\mathsf{M} = m).
\end{aligned}$$

Fix $m \geq 3$. Put

$$C = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \text{There exists } i, j \in \{1, \ldots, m\}, i \neq j \text{ such that } x_i = x_j\}.$$

Furthermore, given that $\mathsf{M} = m$, put

$$\Psi := \{(\mathsf{x}_1, \ldots, \mathsf{x}_m) : \mathsf{x}_1 \in \frac{1}{\rho_{\mathsf{z}'_{(1o)}}}\Phi_{(1)}, \ldots, \mathsf{x}_m \in \frac{1}{\rho_{\mathsf{z}'_{(mo)}}}\Phi_{(m)}\}.$$

Obviously, Ψ is a stationary simple point process in $\mathbb{R}^m$. From the fact that the vector $(\rho_{\mathsf{z}'_{(1o)}}, \ldots, \rho_{\mathsf{z}'_{(mo)}})$ is independent of the vector $(\Phi_{(1)}, \ldots, \Phi_{(m)})$ as well as $\Phi_{(1)}, \ldots, \Phi_{(m)}$ are i.i.d. stationary simple point processes with intensity 1, we find that

$$\begin{aligned}
\mathbb{E}[\Psi(B_1 \times \ldots \times B_m)|\mathsf{M} = m] &= \mathbb{E}\Bigg[\frac{1}{\rho_{\mathsf{z}'_{(1o)}}}\Phi_{(1)}(B_1) \ldots \frac{1}{\rho_{\mathsf{z}'_{(mo)}}}\Phi_{(m)}(B_m)\big|\mathsf{M} = m\Bigg]\\
&= \mathbb{E}\Bigg[\frac{1}{\rho_{\mathsf{z}'_{(1o)}} \ldots \rho_{\mathsf{z}'_{(mo)}}}\Bigg|\mathsf{M} = m\Bigg]\mathbb{E}\Phi_{(1)}(B_1) \ldots \mathbb{E}\Phi_{(m)}(B_m)\\
&= \mathbb{E}\Bigg[\frac{1}{\rho_{\mathsf{z}'_{(1o)}} \ldots \rho_{\mathsf{z}'_{(mo)}}}\Bigg|\mathsf{M} = m\Bigg]\lambda_1(B_1) \ldots \lambda_1(B_m)\\
&= \mathbb{E}\Bigg[\frac{1}{\rho_{\mathsf{z}'_{(1o)}} \ldots \rho_{\mathsf{z}'_{(mo)}}}\Bigg|\mathsf{M} = m\Bigg]\lambda_m(B_1 \times \ldots \times B_m)
\end{aligned}$$

for $B_1, \dots, B_m \in \mathcal{B}(\mathbb{R})$. Here λ_1 and λ_m are the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^m$, respectively. We observe that

$$\begin{aligned}
&\mathbb{P}\Bigg(\text{There exists a non-simple point of } \bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \Bigg| \mathsf{M} = m\Bigg) \\
&= \mathbb{E}\mathbf{1}\Bigg\{\text{There exists a non-simple point of } \bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \Bigg| \mathsf{M} = m\Bigg\} \\
&\le \mathbb{E}\Bigg(\sum_{\mathsf{x} \in \operatorname{supp}\left(\bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}\right)} \mathbf{1}(n(\mathsf{x}) > 1) \Bigg| \mathsf{M} = m\Bigg).
\end{aligned}$$

Here $n(\mathsf{x})$ is the multiplicity of x and $\operatorname{supp}\left(\bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}\right)$ is the support of $\bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}$ regardless the multiplicities of points. We arrive at

$$\begin{aligned}
&\mathbb{P}\Bigg(\text{There exists a non-simple point of } \bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \Bigg| \mathsf{M} = m\Bigg) \\
&\le \mathbb{E}\Bigg(\sum_{(\mathsf{x}_1,\dots,\mathsf{x}_m) \in \Psi} \mathbf{1}_C((\mathsf{x}_1,\dots,\mathsf{x}_m)) \Bigg| \mathsf{M} = m\Bigg) = \mathbb{E}[\Psi(C) | \mathsf{M} = m] \\
&= \mathbb{E}\left[\frac{1}{\rho_{\mathsf{z}'_{(1o)}} \cdots \rho_{\mathsf{z}'_{(mo)}}} \Bigg| \mathsf{M} = m\right] \lambda_m(C) = 0.
\end{aligned}$$

Hence, $\mathbb{P}\left(\bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \text{ is a simple point process on } \mathbb{R} \,\Big|\, \mathsf{M} = m\right) = 1$ for all $m \ge 3$. Consequently

$$\mathbb{P}\left(\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} \text{ is a simple point process on } \mathbb{R}\right) = \sum_{m=3}^{\infty} \mathbb{P}(\mathsf{M} = m) = 1.$$

On the other hand, for any $B \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$ we have, using the simplicity of $\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}$ and the stationary of $\Phi_{(i)}$, $i = 1, 2, \dots, \mathsf{M}$,

$$\begin{aligned}
\left(\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)} + x\right)(B) &= \left(\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}\right)(B - x) = \sum_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}(B - x) \\
&\stackrel{\mathcal{D}}{=} \sum_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}(B) = \left(\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}\right)(B).
\end{aligned}$$

Hence, $\bigcup_{i=1}^{\mathsf{M}} \frac{1}{\rho_{\mathsf{z}'_{(io)}}}\Phi_{(i)}$ is a stationary point process on $\mathbb{R}$. □

Because of Property 2.1.1, there is no coplanarity of cross-sectional plates that appear in neighbouring columns, and so the cells in neighbouring columns cannot have a common facet. So a column tessellation is not facet-to-facet. The intersection

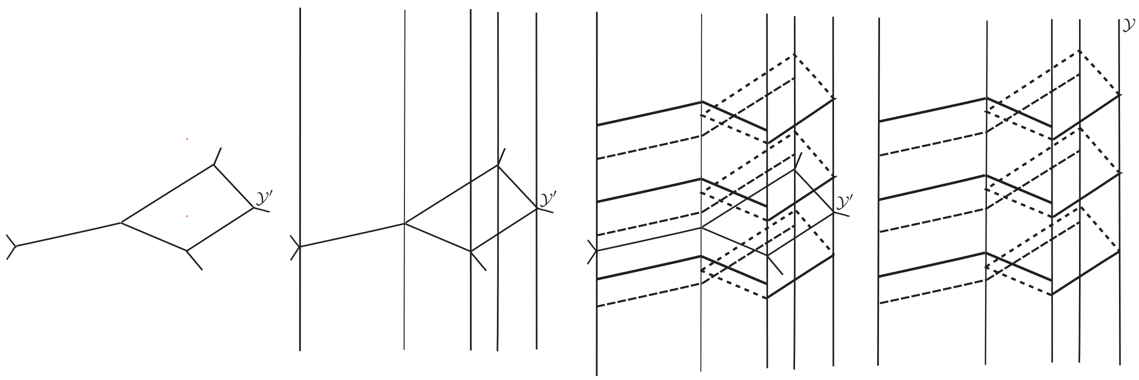

FIGURE 9. Column tessellation $\mathcal{Y}$ with constant height 1. Here the four steps building up a column tessellation are shown: Starting with the planar tessellation $\mathcal{Y}'$, next the columns are formed by the cells of $\mathcal{Y}'$. Then the columns with the cuts generated by the parallel horizontal plates are shown, whereas in the last figure $\mathcal{Y}'$ is removed.

of a column tessellation $\mathcal{Y}$ with any fixed plane parallel to $\mathcal{E}$ is a vertical translation of $\mathcal{Y}'$ almost surely. We emphasize that each cell of the planar tessellation $\mathcal{Y}'$ is not a horizontal plate of the column tessellation $\mathcal{Y}$.

2.1.1. Constant cell heights. For $j = 1, 2, \ldots$ let $\zeta_k^{(j)}$, $k = 0, \pm 1, \pm 2, \ldots$ be the random distances from $\mathcal{E}$ of the horizontal plates in the column based on z'_j. A very simple case arises when $\zeta_0^{(j)}$, $j = 1, 2, \ldots$ is a sequence of i.i.d random variables which are uniformly distributed in $(0, 1]$ and $\zeta_k^{(j)} = \zeta_0^{(j)} + k$ for all k. This implies, of course, that $\rho_{\mathsf{z}'} = 1$, a constant for all cells z' of $\mathcal{Y}'$. Also the positions of the horizontal plates in a column are stationary and *completely* independent of such plates in the neighbouring columns, as no information has been drawn from $\mathcal{Y}'$. Any cell of the column tessellation $\mathcal{Y}$ that has arisen is a right prism with height 1. For short, we call it a *column tessellation with height 1*. An illustration is given in Figure 9.

2.1.2. Notation. In order to work with the multisets Z_0, Z_1, Z_2 and P_1 of the column tessellation $\mathcal{Y}$; see Remarks 1.3.7 and 1.3.8 for the description of these multisets, we need further notation for the random planar tessellation $\mathcal{Y}'$. Some is based on a relationship between cells and lower dimensional objects of the planar tessellation - an *ownership relation*. We describe the ownership relation using a relation b or a function b_π (belonging-to) as follows; see also [**39**].

Denote by $\mathsf{Z}_j'^{\neq}$, $j = 0$ or $j = 1$, the set of all j-dimensional faces of all cells in $\mathcal{Y}'$ but without multiplicity. We emphasize that $\mathsf{Z}_j'^{\neq}$ is not equal to the multiset Z'_j; see the end of Section 1.3 for the description of the multiset Z'_j. For example if $j = 0$ then $\mathsf{Z}_0'^{\neq} = \mathsf{V}'$. Furthermore, let $Z'(y')$ and $Z_j'^{\neq}(y')$ be the set of all cells and the set of all j-dimensional faces of all cells of a fixed realization y' of $\mathcal{Y}'$ respectively.

Definition 2.1.2. An *ownership relation* for y', denoted by $b(y')$, is a subset of $Z_j'^{\neq}(y') \times Z'(y')$ defined by $b(y') := \{(z_j', z') \in Z_j'^{\neq}(y') \times Z'(y') : z_j' \text{ is a } j\text{-face of } z'\}$. The cell z' is called an *owner cell* of z_j' if $(z_j', z') \in b(y')$.

Remark 2.1.3. For the stationary random planar tessellation $\mathcal{Y}'$, $b := b(\mathcal{Y}')$ is a random ownership relation.

It is obvious that z' is the owner cell of $n_j(\mathsf{z}')$ its j-dimensional faces (each of these faces belongs to $\mathsf{Z}_j'^{\neq}$) and that any $\mathsf{z}_j' \in \mathsf{Z}_j'^{\neq}$ can have more than one owner cell. The number of owner cells of an object $\mathsf{z}_j' \in \mathsf{Z}_j'^{\neq}$ is denoted by $n_{\mathsf{Z}'}(\mathsf{z}_j')$. Note that the notation $n_j(\mathsf{z}')$ and $n_{\mathsf{Z}'}(\mathsf{z}_j')$ have different meanings.

Definition 2.1.4. The multisets Z_0' and Z_1' are defined as follows:

$$\mathsf{Z}_0' := \{(\mathsf{v}', n_{\mathsf{Z}'}(\mathsf{v}')) : \mathsf{v}' \in \mathsf{Z}_0'^{\neq} = \mathsf{V}'\}, \quad \mathsf{Z}_1' := \{(\mathsf{s}', n_{\mathsf{Z}'}(\mathsf{s}')) : \mathsf{s}' \in \mathsf{Z}_1'^{\neq}\}.$$

Furthermore we are also interested in the vertices of a cell which are not 0-faces of that cell. It is obvious that those vertices are π-vertices. We say that z' is the π*-owner cell* of such a π-vertex which is not a 0-face of z'. We use for this special belonging-to function the symbol b_π. Thus any π-vertex $\mathsf{v}'[\pi]$ belongs to a unique π-owner-cell $\mathsf{z}' = b_\pi(\mathsf{v}'[\pi])$ and a cell z' is the π-owner-cell of $(m_{\mathsf{V}'}(\mathsf{z}') - n_0(\mathsf{z}'))$ π-vertices. So our belonging-to relation b has domain $\mathsf{V}' \cup \mathsf{Z}_1'^{\neq}$ and codomain Z', whereas function b_π has domain $\mathsf{V}'[\pi]$ – the set of all π-vertices of $\mathcal{Y}'$ – and range Z'.

Later we will investigate the dependency of the vertex intensity of a column tessellation on both the planar tessellation and the given marks $\rho_{\mathsf{z}'}$. It is easy to see that all vertices of $\mathcal{Y}$ are located on vertical lines through the vertices of $\mathcal{Y}'$. Also the intensity of the vertices on such a vertex-line depends on the ρ-intensity of all the planar cells adjacent to the planar vertex which creates this vertex-line. To describe those relations between $\mathcal{Y}$, $\mathcal{Y}'$ and the cell marks $\rho_{\mathsf{z}'}$ we define further entities for the planar tessellation $\mathcal{Y}'$ as follows.

- For a fixed x' and adjacency relation,
  $$\alpha_{\mathsf{x}'} = \sum_{\{\mathsf{z}':\mathsf{z}'\supset\mathsf{x}'\}} \rho_{\mathsf{z}'},$$
  where we mostly consider the cases $\mathsf{x}' = \mathsf{v}' \in \mathsf{V}'$, $\mathsf{x}' = \mathsf{e}' \in \mathsf{E}'$, $\mathsf{x}' = \mathsf{v}'[\pi] \in \mathsf{V}'[\pi]$ and $\mathsf{x}' = \mathsf{v}'[\bar{\pi}] \in \mathsf{V}'[\bar{\pi}]$ – the set of all non-π-vertices of $\mathcal{Y}'$,
- For a fixed $\mathsf{z}_0' = \mathsf{v}' \in \mathsf{V}' = \mathsf{Z}_0'^{\neq}$ or $\mathsf{z}_1' \in \mathsf{Z}_1'^{\neq}$ or $\mathsf{v}'[\pi] \in \mathsf{V}'[\pi]$ and ownership relation,
  $$\beta_{\mathsf{z}_j'} = \sum_{\{\mathsf{z}':(\mathsf{z}_j',\mathsf{z}')\in b\}} \rho_{\mathsf{z}'},\ j = 0 \text{ or } j = 1,$$
  $$\epsilon_{\mathsf{v}'[\pi]} = \rho_{b_\pi(\mathsf{v}'[\pi])},$$
  and
- with *weighting*,
  $\theta_{\mathsf{v}'} = m_{\mathsf{E}'}(\mathsf{v}')\alpha_{\mathsf{v}'}$, (number-weighted)
  $\theta_{\mathsf{e}'} = \ell(\mathsf{e}')\alpha_{\mathsf{e}'}$, (length-weighted)
  $\theta_{\mathsf{z}'} = a(\mathsf{z}')\rho_{\mathsf{z}'}$, (area-weighted)

  where $\ell(\mathsf{e}')$ is the length of the edge e' and $a(\mathsf{z}')$ is the area of the cell z'.

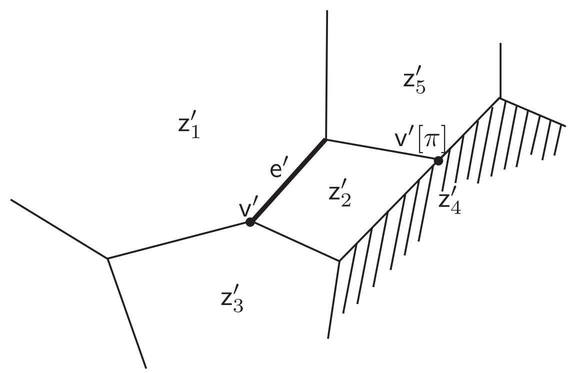

FIGURE 10. An example of adjacency and ownership relations in the planar tessellation $\mathcal{Y}'$.

Figure 10 illustrates an example that shows our notation and also the differences between ownership and adjacency relations. The vertex v' is adjacent to the cells z'_1, z'_2 and z'_3, the edge e' is adjacent to the cells z'_1 and z'_2, hence $\alpha_{\mathsf{v}'} = \rho_{\mathsf{z}'_1} + \rho_{\mathsf{z}'_2} + \rho_{\mathsf{z}'_3}$ and $\alpha_{\mathsf{e}'} = \rho_{\mathsf{z}'_1} + \rho_{\mathsf{z}'_2}$. For the ownership relation, it is easy to see that for the π-vertex $\mathsf{v}'[\pi]$ we have $\epsilon_{\mathsf{v}'[\pi]} = \rho_{\mathsf{z}'_4}$, because z'_4 is the π-owner cell of $\mathsf{v}'[\pi]$. Besides, v' is a 0-face of the three cells z'_1, z'_2 and z'_3, that is, v' has three owner cells z'_1, z'_2 and z'_3. Hence $\beta_{\mathsf{v}'} = \rho_{\mathsf{z}'_1} + \rho_{\mathsf{z}'_2} + \rho_{\mathsf{z}'_3}$. While $\mathsf{v}'[\pi]$ is a 0-face of the two cells z'_2 and z'_5, that is, $\mathsf{v}'[\pi]$ has two owner cells z'_2 and z'_5. Hence $\beta_{\mathsf{v}'[\pi]} = \rho_{\mathsf{z}'_2} + \rho_{\mathsf{z}'_5}$.

Later we will see that the last two items of the new notation, $\theta_{\mathsf{e}'}$ and $\theta_{\mathsf{z}'}$, are necessary for the studies of the metrical properties whereas all others are used for results concerning topological and interior parameters. Recall now Definition 1.3.15 of the Palm distribution of the stationary random planar tessellation $\mathcal{Y}'$ with respect to the typical X'-type object.

Definition 2.1.5. For $\mathsf{X}' \in \{\mathsf{V}', \mathsf{V}'[\pi], \mathsf{V}'[\bar{\pi}], \mathsf{Z}_1'^{\neq}, \mathsf{E}', \mathsf{Z}'\}$ the Palm distribution of the stationary planar tessellation $\mathcal{Y}'$ with respect to the typical X'-type object is the probability measure $\mathbb{Q}_{\mathsf{X}'}$ on $\mathscr{T}'$ given by

$$\mathbb{Q}_{\mathsf{X}'}(A) := \frac{1}{\gamma_{\mathsf{X}'}\lambda_2(B)} \mathbb{E} \sum_{\mathsf{x}' \in \mathsf{X}'} \mathbf{1}_B(c(\mathsf{x}'))\mathbf{1}_A(\mathcal{Y}' - c(\mathsf{x}')) \tag{22}$$

for $A \in \mathcal{T}'$ and $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$. Denote by $\mathbb{E}_{\mathsf{X}'}$ the expectation with respect to $\mathbb{Q}_{\mathsf{X}'}$. From now $\mathbb{Q}_{\mathsf{X}'}$ and $\mathbb{E}_{\mathsf{X}'}$ are only understood in this sense. The grain distributions of particle processes in $\mathbb{R}^2$ as well as the mark distributions of marked point processes in $\mathbb{R}^2$ with some mark space have another notation which will be introduced later for convenience.

Definition 2.1.6. The means of α-, β-, ϵ-, θ-quantities in the random context are as follows:

- $\overline{\rho}_{\mathsf{Z}'} := \mathbb{E}_{\mathsf{Z}'}(\rho_{\mathsf{z}'})$ – the mean ρ-intensity of the typical cell,
- $\overline{\alpha}_{\mathsf{X}'} := \mathbb{E}_{\mathsf{X}'}(\alpha_{\mathsf{x}'})$ – the mean total ρ-intensity of all cells adjacent to the typical X'-type object, $\mathsf{X}' \in \{\mathsf{V}', \mathsf{V}'[\pi], \mathsf{V}'[\bar{\pi}], \mathsf{E}'\}$,

- $\overline{\beta}_{\mathsf{V}'} := \mathbb{E}_{\mathsf{V}'}(\beta_{\mathsf{v}'})$ – the mean total ρ-intensity of all owner cells of the typical vertex,
- $\bar{\epsilon}_{\mathsf{V}'[\pi]} := \mathbb{E}_{\mathsf{V}'[\pi]}(\epsilon_{\mathsf{v}'[\pi]})$ – the mean ρ-intensity of the π-owner cell of the typical π-vertex and
- $\overline{\theta}_{\mathsf{X}'} := \mathbb{E}_{\mathsf{X}'}(\theta_{\mathsf{x}'})$ – the mean total weighted ρ-intensity of all cells adjacent to the typical X'-type object, $\mathsf{X}' \in \{\mathsf{V}', \mathsf{E}', \mathsf{Z}'\}$.

Using mean value identities for the primitive tessellation elements or subsets given in [**39**], most of the above mean values can be expressed as second-order quantities depending on the ρ-intensity as follows:

Lemma 2.1.7. *We have*

(i) $\gamma_{\mathsf{X}'}\overline{\alpha}_{\mathsf{X}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(m_{\mathsf{X}'}(\mathsf{z}')\rho_{\mathsf{z}'})$,

(ii) $\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(n_0(\mathsf{z}')\rho_{\mathsf{z}'})$,

(iii) $\gamma_{\mathsf{V}'[\pi]}\bar{\epsilon}_{\mathsf{V}'[\pi]} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}[(m_{\mathsf{V}'}(\mathsf{z}') - n_0(\mathsf{z}'))\rho_{\mathsf{z}'}] = \gamma_{\mathsf{V}'}\bar{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\bar{\beta}_{\mathsf{V}'}$,

(iv) $\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\ell(\mathsf{z}')\rho_{\mathsf{z}'})$, *where* $\ell(\mathsf{z}')$ *is the perimeter of the planar cell* z',

(v) $\gamma_{\mathsf{Z}'}\overline{\theta}_{\mathsf{Z}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(a(\mathsf{z}')\rho_{\mathsf{z}'})$.

(vi) $\gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(k_{\mathsf{E}'}(\mathsf{z}')\rho_{\mathsf{z}'}) + \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$, *where* $k_{\mathsf{E}'}(\mathsf{z}') := \sum_{\mathsf{e}' \in \mathsf{E}'} \mathbf{1}\{\mathsf{e}' \cap \mathsf{z}' \neq \emptyset\}$ *is the number of edges intersecting* z'. *More particularly,* $k_{\mathsf{E}'}(\mathsf{z}')$ *is the sum of the number of edges on the boundary of* z' *and the number of edges which intersect* z' *at some vertex on the boundary of* z'.

Proof.

(i) By definition $\gamma_{\mathsf{X}'}\overline{\alpha}_{\mathsf{X}'} = \gamma_{\mathsf{X}'}\mathbb{E}_{\mathsf{X}'}(\alpha_{\mathsf{x}'}) = \gamma_{\mathsf{X}'}\mathbb{E}_{\mathsf{X}'}\big(\sum_{\{\mathsf{z}':\mathsf{z}'\supset\mathsf{x}'\}} \rho_{\mathsf{z}'}\big)$. Using mean value identities given in [**39**] at this step, the latter is $\gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\big(\rho_{\mathsf{z}'} \sum_{\{\mathsf{x}':\mathsf{x}'\subset\mathsf{z}'\}} 1\big)$, namely $\gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\rho_{\mathsf{z}'} m_{\mathsf{X}'}(\mathsf{z}'))$.

(ii) We have

$$\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}\Big(\sum_{\{\mathsf{z}':(\mathsf{v}',\mathsf{z}')\in b\}} \rho_{\mathsf{z}'}\Big) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\Big(\rho_{\mathsf{z}'} \sum_{\{\mathsf{v}':(\mathsf{v}',\mathsf{z}')\in b\}} 1\Big) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\rho_{\mathsf{z}'} n_0(\mathsf{z}')).$$

(iii) We find that

$$\begin{aligned}\gamma_{\mathsf{V}'[\pi]}\bar{\epsilon}_{\mathsf{V}'[\pi]} &= \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}(\rho_{b_\pi(\mathsf{v}'[\pi])}) = \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}\Big(\sum_{\{\mathsf{z}':\mathsf{z}'=b_\pi(\mathsf{v}'[\pi])\}} \rho_{\mathsf{z}'}\Big)\\ &= \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\Big(\rho_{\mathsf{z}'} \sum_{\{\mathsf{v}'[\pi]:\mathsf{z}'=b_\pi(\mathsf{v}'[\pi])\}} 1\Big) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}[\rho_{\mathsf{z}'}(m_{\mathsf{V}'}(\mathsf{z}') - n_0(\mathsf{z}'))]\\ &= \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.\end{aligned}$$

(iv) We have

$$\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} = \gamma_{\mathsf{E}'}\mathbb{E}_{\mathsf{E}'}\Big(\ell(\mathsf{e}') \sum_{\{\mathsf{z}':\mathsf{z}'\supset\mathsf{e}'\}} \rho_{\mathsf{z}'}\Big) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\Big(\rho_{\mathsf{z}'} \sum_{\{\mathsf{e}':\mathsf{e}'\subset\mathsf{z}'\}} \ell(\mathsf{e}')\Big) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\ell(\mathsf{z}')\rho_{\mathsf{z}'}).$$

(v) By definition $\gamma_{\mathsf{Z}'}\overline{\theta}_{\mathsf{Z}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\theta_{\mathsf{z}'}) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(a(\mathsf{z}')\rho_{\mathsf{z}'})$.

(vi) By definition $\gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')\alpha_{\mathsf{v}'}) = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}\big(m_{\mathsf{E}'}(\mathsf{v}')\sum_{\{\mathsf{z}':\mathsf{z}'\supset\mathsf{v}'\}}\rho_{\mathsf{z}'}\big)$. The latter is $\gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\big(\rho_{\mathsf{z}'}\sum_{\{\mathsf{v}':\mathsf{v}'\subset\mathsf{z}'\}} m_{\mathsf{E}'}(\mathsf{v}')\big)$. On the other hand, we observe that

$$\gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\Big(\rho_{\mathsf{z}'}\sum_{\{\mathsf{v}':\mathsf{v}'\subset\mathsf{z}'\}} m_{\mathsf{E}'}(\mathsf{v}')\Big) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}[\rho_{\mathsf{z}'}(k_{\mathsf{E}'}(\mathsf{z}')+m_{\mathsf{V}'}(\mathsf{z}'))] = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(k_{\mathsf{E}'}(\mathsf{z}')\rho_{\mathsf{z}'})+\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}.$$

□

Remark 2.1.8. To illustrate these mean values we consider now the special case when $\rho_{\mathsf{z}'} = 1$ for all $\mathsf{z}' \in \mathsf{Z}'$.

$$\overline{\rho}_{\mathsf{Z}'} = 1, \quad \overline{\alpha}_{\mathsf{V}'} = \mu_{\mathsf{V}'\mathsf{Z}'} = \mu_{\mathsf{V}'\mathsf{E}'}, \quad \overline{\alpha}_{\mathsf{E}'} = 2, \quad \overline{\alpha}_{\mathsf{V}'[\pi]} = \mu_{\mathsf{V}'[\pi]\mathsf{Z}'} = \mu_{\mathsf{V}'[\pi]\mathsf{E}'},$$

$$\overline{\beta}_{\mathsf{V}'} = \nu_{\mathsf{V}'\mathsf{Z}'} = \frac{\gamma_{\mathsf{V}'[\pi]}(\mu_{\mathsf{V}'[\pi]\mathsf{E}'} - 1) + \gamma_{\mathsf{V}'[\bar{\pi}]}\mu_{\mathsf{V}'[\bar{\pi}]\mathsf{E}'}}{\gamma_{\mathsf{V}'}} = \frac{\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma_{\mathsf{V}'[\pi]}}{\gamma_{\mathsf{V}'}} = \mu_{\mathsf{V}'\mathsf{E}'} - \phi,$$

$$\overline{\epsilon}_{\mathsf{V}'[\pi]} = 1, \quad \overline{\theta}_{\mathsf{V}'} = \mu^{(2)}_{\mathsf{V}'\mathsf{E}'}, \quad \overline{\theta}_{\mathsf{E}'} = 2\overline{\ell}_{\mathsf{E}'}, \quad \overline{\theta}_{\mathsf{Z}'} = \bar{a}_{\mathsf{Z}'},$$

where

$\mu_{\mathsf{V}'[\pi]\mathsf{E}'}$	–	the mean number of emanating edges from the typical π-vertex,
$\nu_{\mathsf{V}'\mathsf{Z}'}$	–	the mean number of owner cells of the typical vertex,
$\mu^{(2)}_{\mathsf{V}'\mathsf{E}'}$	–	the second moment of the number of edges adjacent to the typical vertex,
$\overline{\ell}_{\mathsf{E}'}$	–	the mean length of the typical edge,
$\bar{a}_{\mathsf{Z}'}$	–	the mean area of the typical cell.

Formally, the second moment of the number of edges adjacent to the typical vertex and the mean number of owner cells of the typical vertex are given by $\mathbb{E}_{\mathsf{V}'}[m_{\mathsf{E}'}(\mathsf{v}')^2]$ and $\mathbb{E}_{\mathsf{V}'}[n_{\mathsf{Z}'}(\mathsf{v}')]$, respectively.

The first and the last formulae in Remark 2.1.8 are obvious. The formulae for the three α-means and β- mean follow from $\alpha_{\mathsf{x}'} = m_{\mathsf{Z}'}(\mathsf{x}')$ and $\beta_{\mathsf{v}'} = n_{\mathsf{Z}'}(\mathsf{v}')$ in the case $\rho_{\mathsf{z}'} = 1$. The ϵ-mean value formula arises from the fact that the π-owner cell of the typical π-vertex has ρ-intensity 1. And the first two θ-mean values we obtain using $m_{\mathsf{Z}'}(\mathsf{v}') = m_{\mathsf{E}'}(\mathsf{v}')$ and again $\alpha_{\mathsf{x}'} = m_{\mathsf{Z}'}(\mathsf{x}')$ for $\mathsf{x}' = \mathsf{v}'$ and $\mathsf{x}' = \mathsf{e}'$:

$$\overline{\theta}_{\mathsf{V}'} = \mathbb{E}_{\mathsf{V}'}(\theta_{\mathsf{v}'}) = \mathbb{E}_{\mathsf{V}'}[m_{\mathsf{E}'}(\mathsf{v}')^2] = \mu^{(2)}_{\mathsf{V}'\mathsf{E}'}$$

and

$$\overline{\theta}_{\mathsf{E}'} = \mathbb{E}_{\mathsf{E}'}(\theta_{\mathsf{e}'}) = \mathbb{E}_{\mathsf{E}'}(\ell(\mathsf{e}')m_{\mathsf{Z}'}(\mathsf{e}')) = \mathbb{E}_{\mathsf{E}'}(2\ell(\mathsf{e}')) = 2\overline{\ell}_{\mathsf{E}'}.$$

Considering again the general construction, our aim is the calculation of intensities and mean values of the column tessellation $\mathcal{Y}$ from the characteristics of $\mathcal{Y}'$. For this purpose the following basic interrelationship between vertices and edges of $\mathcal{Y}$ and $\mathcal{Y}'$ are helpful.

2.1.3. Basic properties. For a vertex $\mathsf{v}' \in \mathsf{V}'$ in $\mathcal{Y}'$ we consider the vertical line $\mathcal{L}_{\mathsf{v}'}$ through v' (called vertex-line) and the columns created by the planar cells adjacent to v'. The horizontal plates in these columns create a point process (comprising vertices of the spatial tessellation $\mathcal{Y}$) on $\mathcal{L}_{\mathsf{v}'}$. This point process, denoted by $\Phi_{\mathsf{v}'}$, is the superposition of stationary simple point processes with ρ-intensities from the planar cells adjacent to v'. Because of Property 2.1.1, it has intensity $\alpha_{\mathsf{v}'}$. Each point of $\Phi_{\mathsf{v}'}$ accepts v' as its corresponding vertex in $\mathcal{Y}'$.

If v' is a non-π-vertex of $\mathcal{Y}'$, denoted by $\mathsf{v}'[\bar{\pi}]$, then each point of $\Phi_{\mathsf{v}'[\bar{\pi}]}$ is the apex (0-face) of 2 cells and in the relative interior of $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])-1$ ridges of $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])-1$ other cells, i.e. adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])+1$ cells. Besides, each point of $\Phi_{\mathsf{v}'[\bar{\pi}]}$ is also the 0-face of 1 horizontal plate and 4 vertical plates and in the relative interior of $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])-2$ sides of $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])-2$ other vertical plates, i.e. adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])+3$ plates.

If v' is a π-vertex, denoted by $\mathsf{v}'[\pi]$, then the horizontal plates in the column based on the cell $b_\pi(\mathsf{v}'[\pi])$ create on $\mathcal{L}_{\mathsf{v}'[\pi]}$ non-hemi-vertices of the column tessellation $\mathcal{Y}$. Hence, the intensity of non-hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is $\epsilon_{\mathsf{v}'[\pi]} = \rho_{b_\pi(\mathsf{v}'[\pi])}$, which implies that the intensity of hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is $\alpha_{\mathsf{v}'[\pi]} - \epsilon_{\mathsf{v}'[\pi]}$. Each point of $\Phi_{\mathsf{v}'[\pi]}$ which is a non-hemi-vertex of $\mathcal{Y}$ is in the relative interior of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+1$ ridges of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+1$ cells, moreover, is the 0-face of 4 vertical plates and in the relative interior of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-1$ sides of 1 horizontal plate and $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ vertical plates, i.e. adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+1$ cells and $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+3$ plates. Whereas each point of $\Phi_{\mathsf{v}'[\pi]}$ which is a hemi-vertex of $\mathcal{Y}$ is the apex of 2 cells, in the relative interior of 1 facet of 1 cell and in the relative interior of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ ridges of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ other cells, moreover, is the 0-face of 1 horizontal plate and 4 vertical plates and in the relative interior of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ sides of $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ other vertical plates, i.e. also adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+1$ cells and $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+3$ plates. Here, a side of some plate is a 1-dimensional face of that plate (a plate-side).

We emphasize that Property 2.1.1 plays a central role in calculating the number of adjacent cells, the number of adjacent plates as well as the number of adjacent relative ridge-interiors and the number of adjacent relative plate-side-interiors of a point of $\Phi_{\mathsf{v}'}$ for any vertex v' in the stationary random planar tessellation $\mathcal{Y}'$. We summarize our remarks in the next property.

Property 2.1.9. Let v' be a vertex in $\mathcal{Y}'$. Then the point process $\Phi_{\mathsf{v}'}$ has intensity $\alpha_{\mathsf{v}'}$ and each point of $\Phi_{\mathsf{v}'}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}')+1$ cells and to $m_{\mathsf{E}'}(\mathsf{v}')+3$ plates of $\mathcal{Y}$. Moreover, if v' is a π-vertex $\mathsf{v}'[\pi]$ then the intensities of non-hemi-vertices and hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is $\epsilon_{\mathsf{v}'[\pi]}$ and $\alpha_{\mathsf{v}'[\pi]} - \epsilon_{\mathsf{v}'[\pi]}$, respectively. Each of the non-hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi])+1$ relative ridge-interiors and $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-1$ relative plate-side-interiors. Each of the hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ relative ridge-interiors and $m_{\mathsf{E}'}(\mathsf{v}'[\pi])-2$ relative plate-side-interiors. If the vertex v' of $\mathcal{Y}'$ is a non-π-vertex $\mathsf{v}'[\bar{\pi}]$, each point of $\Phi_{\mathsf{v}'[\bar{\pi}]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])-1$ relative ridge-interiors and $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])-2$ relative plate-side-interiors.

Furthermore the column tessellation has only horizontal and vertical edges. All horizontal edges are π-edges with three emanating plates. For each edge e' of $\mathcal{Y}'$, we have two planar cells adjacent to this edge. When we cut the two corresponding columns by different horizontal planes, we obtain a particle process of horizontal edges of $\mathcal{Y}$ in the common face of the two neighbouring columns. This particle process, denoted by $\Phi_{\mathsf{e}'}$, has intensity $\alpha_{\mathsf{e}'}$. All the edges of $\Phi_{\mathsf{e}'}$ are translations of e'. Besides, for any $\mathsf{v}' \in \mathsf{V}'$, the intensity of the particle process of vertical edges of $\mathcal{Y}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'}$ is $\alpha_{\mathsf{v}'}$ (the reference point of a vertical edge of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'}$ is chosen to be its lower endpoint, which is a point of $\Phi_{\mathsf{v}'}$).

Property 2.1.10. Let e' be an edge in $\mathcal{Y}'$. Then the particle process $\Phi_{\mathsf{e}'}$ has intensity $\alpha_{\mathsf{e}'}$.
Any horizontal edge of $\mathcal{Y}$ is a π-edge with three emanating plates, two of them are vertical, the third one is a horizontal plate. Any horizontal edge of $\mathcal{Y}$ is adjacent to three cells.
Let v' be a vertex in $\mathcal{Y}'$. The intensity of the particle process of vertical edges of $\mathcal{Y}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'}$ is $\alpha_{\mathsf{v}'}$. Each particle (vertical edge) of this process accepts v' as its corresponding vertex in $\mathcal{Y}'$ and is adjacent to $m_{\mathsf{E}'}(\mathsf{v}')$ cells and $m_{\mathsf{E}'}(\mathsf{v}')$ plates of $\mathcal{Y}$.

2.2. Formulae for the features of column tessellations

2.2.1. Intensities of primitive elements.

As a first step we will consider how the intensities γ_{X} of the primitive elements $\mathsf{X} \in \{\mathsf{V}, \mathsf{E}, \mathsf{P}, \mathsf{Z}\}$ of a column tessellation $\mathcal{Y}$ depend on characteristics of the random planar tessellation $\mathcal{Y}'$. To establish formulae for this dependence we need the intensities $\gamma_{\mathsf{Z}'}$ and $\gamma_{\mathsf{V}'}$, the mean ρ-intensity $\overline{\rho}_{\mathsf{Z}'}$ and the mean total ρ-intensity $\overline{\alpha}_{\mathsf{V}'}$ of $\mathcal{Y}'$:

Proposition 2.2.1. *The intensities of primitive elements of the column tessellation $\mathcal{Y}$ depend on $\mathcal{Y}'$ and the cell marks $\rho_{\mathsf{z}'}$ as follows:*

(i) $\gamma_{\mathsf{Z}} = \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}$,
(ii) $\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$,
(iii) $\gamma_{\mathsf{E}} = 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$,
(iv) $\gamma_{\mathsf{P}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}$.

For a refined partition of the sets E and P of $\mathcal{Y}$ into horizontal and vertical elements we obtain

(v) $\gamma_{\mathsf{E}[\mathsf{hor}]} = \gamma_{\mathsf{E}[\mathsf{vert}]} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$ *and*
(vi) $\gamma_{\mathsf{P}[\mathsf{hor}]} = \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}$, $\gamma_{\mathsf{P}[\mathsf{vert}]} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$.

Proof. (i) Recall that Φ_j, $j = 1, 2, \ldots$ is a sequence of i.i.d. stationary simple point processes on $\{0\}^2 \times \mathbb{R}$ with intensity 1. The point processes $\Phi_1, \Phi_2, \ldots$ and the tessellation $\mathcal{Y}' = \{\mathsf{z}'_j : j = 1, 2, \ldots\}$ are independent. Let $\mathbb{Q}_1$ be the distribution of Φ_1.

Note that $\mathsf{z}'_{jo} = \mathsf{z}'_j - c(\mathsf{z}'_j)$ has circumcenter at the origin o. For each cell z'_j of the stationary random planar tessellation $\mathcal{Y}'$, we mark its circumcenter $c(\mathsf{z}'_j)$ with z'_{jo} and Φ_j. We obtain a marked point process, denoted by $\widetilde{\Phi}$, in the product space

$(\mathbb{R}^2 \times \{0\}) \times \mathscr{P}_2^o \times \mathscr{N}_s(\{0\}^2 \times \mathbb{R})$, which from now on we write as $\mathbb{R}^2 \times \mathscr{P}_2^o \times \mathscr{N}_s$ for short. Note that the column tessellation $\mathcal{Y}$ is uniquely determined by $\widetilde{\Phi}$.

Denote by $\mathbb{P}_{\mathcal{Y}}$ the distribution of the column tessellation $\mathcal{Y}$ and $Z(y)$ the set of all cells of a realization y of $\mathcal{Y}$. Moreover, let $r(z)$ be the reference point of the spatial cell z which is the circumcenter of the base facet of z. In order to calculate the cell-intensity of $\mathcal{Y}$, for each cell z'_j of $\mathcal{Y}'$ we consider the cell-axis $\mathcal{L}_{c(\mathsf{z}'_j)}$ (the vertical line going through $c(\mathsf{z}'_j)$). The horizontal plates in the column based on z'_j form a point process on $\mathcal{L}_{c(\mathsf{z}'_j)}$ (comprising cell-circumcenters of $\mathcal{Y}$). Consequently, if $\mathbb{P}_{\widetilde{\Phi}}$ denotes the distribution of the marked point process $\widetilde{\Phi}$, we find that

$$\gamma_{\mathsf{Z}} = \int \sum_{z \in Z(y)} \mathbf{1}_{[0,1]^3}(r(z)) \mathbb{P}_{\mathcal{Y}}(\mathrm{d}y) = \int \sum_{(c(z'_j), z'_{jo}, \varphi_j) \in \widetilde{\varphi}} \; \sum_{c \in \frac{1}{\rho_{z'_{jo}}} \varphi_j + c(z'_j)} \mathbf{1}_{[0,1]^3}(c) \mathbb{P}_{\widetilde{\Phi}}(\mathrm{d}\widetilde{\varphi})$$

$$= \int\limits_{\mathbb{R}^2 \times \mathscr{P}_2^o \times \mathscr{N}_s} \; \sum_{c \in \frac{1}{\rho_{z'_o}} \varphi + c'} \mathbf{1}_{[0,1]^3}(c) \widetilde{\Theta}(\mathrm{d}(c', z'_o, \varphi)),$$

where $\widetilde{\Theta}$ is the intensity measure of the marked point process $\widetilde{\Phi}$. This leads to, using Theorem 1.1.15,

$$\gamma_{\mathsf{Z}} = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \int\limits_{\mathbb{R}^2} \sum_{c \in \frac{1}{\rho_{z'_o}} \varphi + c'} \mathbf{1}_{[0,1]^3}(c) \lambda_2(\mathrm{d}c') \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)).$$

Here $\widetilde{\mathbb{Q}}$ is the mark distribution of $\widetilde{\Phi}$. Besides $\widetilde{\mathbb{Q}}$ is the joint distribution of the two marks of the typical cell-circumcenter in $\mathcal{Y}'$. Denote these two marks by z'_o and Φ. Write $c := (c_1, c_2, c_3)$. Then

$$\begin{aligned}
\gamma_{\mathsf{Z}} &= \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \int\limits_{\mathbb{R}^2} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}} \varphi} \mathbf{1}_{[0,1]^2}((c_1, c_2)) \mathbf{1}_{[0,1]}(c_3) \lambda_2(\mathrm{d}(c_1, c_2)) \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)) \\
&= \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \int\limits_{\mathbb{R}^2} \mathbf{1}_{[0,1]^2}((c_1, c_2)) \Bigg(\sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}} \varphi} \mathbf{1}_{[0,1]}(c_3) \Bigg) \lambda_2(\mathrm{d}(c_1, c_2)) \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)) \\
&= \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \Bigg(\sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}} \varphi} \mathbf{1}_{[0,1]}(c_3) \Bigg) \int\limits_{\mathbb{R}^2} \mathbf{1}_{[0,1]^2}((c_1, c_2)) \lambda_2(\mathrm{d}(c_1, c_2)) \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)) \\
&= \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \Bigg(\sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}} \varphi} \mathbf{1}_{[0,1]}(c_3) \Bigg) \lambda_2([0,1]^2) \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)) \\
&= \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}} \varphi} \mathbf{1}_{[0,1]}(c_3) \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)).
\end{aligned}$$

The mark Φ is a stationary simple point process on $\{0\}^2 \times \mathbb{R}$ with intensity 1 and distribution $\mathbb{Q}_1$. Let $\mathbb{Q}'_1$ be the grain distribution of Z'. Since Φ is independent of

$\mathcal{Y}'$, we infer that

$$\gamma_{\mathsf{Z}} = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o} \int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}} \varphi} \mathbf{1}_{[0,1]}(c_3) \mathbb{Q}_1(\mathrm{d}\varphi) \mathbb{Q}'_1(\mathrm{d}z'_o) = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o} \rho_{z'_o} \mathbb{Q}'_1(\mathrm{d}z'_o).$$

On the other hand, for any planar tessellation T', let $z'_o(T')$ be the cell of T' with circumcenter at the origin o if such a cell exists. Otherwise, let $z'_o(T') = \emptyset$. Moreover, we choose $A = \{T' \in \mathscr{T}' : z'_o(T') \in C\}$, where C is a Borel subset of $\mathscr{P}_2^o$. We have, for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\begin{aligned}
\int\limits_{\mathscr{T}'} \mathbf{1}_C(z'_o(T')) \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') &= \int\limits_{\mathscr{T}'} \mathbf{1}_A(T') \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') = \frac{1}{\gamma_{\mathsf{Z}'} \lambda_2(B)} \mathbb{E} \sum_{\mathsf{z}' \in \mathsf{Z}',\, c(\mathsf{z}') \in B} \mathbf{1}_A(\mathcal{Y}' - c(\mathsf{z}')) \\
&= \frac{1}{\gamma_{\mathsf{Z}'} \lambda_2(B)} \mathbb{E} \sum_{\mathsf{z}' \in \mathsf{Z}'} \mathbf{1}_B(c(\mathsf{z}')) \mathbf{1}_C(z'_o(\mathcal{Y}' - c(\mathsf{z}'))) \\
&= \frac{1}{\gamma_{\mathsf{Z}'} \lambda_2(B)} \mathbb{E} \sum_{\mathsf{z}' \in \mathsf{Z}'} \mathbf{1}_B(c(\mathsf{z}')) \mathbf{1}_C(\mathsf{z}' - c(\mathsf{z}')) = \int\limits_{\mathscr{P}_2^o} \mathbf{1}_C(z'_o) \mathbb{Q}'_1(\mathrm{d}z'_o).
\end{aligned}$$

By a standard argument of integration theory, we get

$$\gamma_{\mathsf{Z}} = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o} \rho_{z'_o} \mathbb{Q}'_1(\mathrm{d}z'_o) = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{T}'} \rho_{z'_o(T')} \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') = \gamma_{\mathsf{Z}'} \mathbb{E}_{\mathsf{Z}'}(\rho_{\mathsf{z}'}) = \gamma_{\mathsf{Z}'} \overline{\rho}_{\mathsf{Z}'}.$$

(ii) For a vertex v'_j of $\mathcal{Y}'$, put $\mathsf{M}_j := m_{\mathsf{Z}'}(\mathsf{v}'_j) = m_{\mathsf{E}'}(\mathsf{v}'_j)$ and denote by $\mathsf{z}'_{j1}, \dots, \mathsf{z}'_{j\mathsf{M}_j}$ the adjacent cells of v'_j. We mark v'_j with its shifted adjacent cells $\mathsf{z}'_{j1} - \mathsf{v}'_j, \dots, \mathsf{z}'_{j\mathsf{M}_j} - \mathsf{v}'_j$, the corresponding i.i.d. stationary simple point processes $\Phi_{j1}, \dots, \Phi_{j\mathsf{M}_j}$ and the random number of its adjacent cells M_j. We obtain a marked point process, denoted by $\widehat{\Phi}$. Note that the set of vertices of the column tessellation $\mathcal{Y}$, namely, V, is uniquely determined by $\widehat{\Phi}$. Denote by $\mathbb{P}_{\mathsf{V}}$ the distribution of V and $\mathbb{P}_{\widehat{\Phi}}$ the distribution of $\widehat{\Phi}$, respectively. Since for any $\mathsf{v}'_j \in \mathsf{V}'$, the process of vertices of $\mathcal{Y}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'_j}$ is the point process $\Phi_{\mathsf{v}'_j}$ (see Subsection 2.1.3), we get

$$\begin{aligned}
\gamma_{\mathsf{V}} &= \int \sum_{v \in \psi} \mathbf{1}_{[0,1]^3}(v) \mathbb{P}_{\mathsf{V}}(\mathrm{d}\psi) \\
&= \int \sum_{(v'_j, z'_{j1} - v'_j, \dots, z'_{jm_j} - v'_j, \varphi_{j1}, \dots, \varphi_{jm_j}, m_j) \in \widehat{\varphi}} \quad \sum_{v \in \bigcup_{i=1}^{m_j} \frac{1}{\rho_{z'_{ji} - v'_j}} \varphi_{ji} + v'_j} \mathbf{1}_{[0,1]^3}(v) \mathbb{P}_{\widehat{\Phi}}(\mathrm{d}\widehat{\varphi}) \\
&= \int \sum_{v \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)} + v'} \mathbf{1}_{[0,1]^3}(v) \widehat{\Theta}(\mathrm{d}(v', z'_{(1)}, \dots, z'_{(m)}, \varphi_{(1)}, \dots, \varphi_{(m)}, m)),
\end{aligned}$$

where $\widehat{\Theta}$ is the intensity measure of the marked point process $\widehat{\Phi}$ and $\bigcup_{i=1}^{m_j} \frac{1}{\rho_{z'_{ji}-v'_j}} \varphi_{ji}$ is the superposition of m_j realizations $\frac{1}{\rho_{z'_{j1}-v'_j}} \varphi_{j1}, \ldots, \frac{1}{\rho_{z'_{jm_j}-v'_j}} \varphi_{jm_j}$.

Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{V}'}$ the vertex at the origin o has exactly m emanating edges, the joint conditional distribution of the first $2m$ marks $\mathsf{z}'_{(1)}, \ldots, \mathsf{z}'_{(m)}, \Phi_{(1)}, \ldots, \Phi_{(m)}$ of the vertex at o is denoted by $\widehat{\mathbb{Q}}$. By definition for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m)$$

$$= \frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)} \mathbb{E} \sum_{\mathsf{v}' \in \mathsf{V}'} \mathbf{1}_B(\mathsf{v}') \mathbf{1}\{m_{\mathsf{E}'}(\mathsf{v}') = m\} = \frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{v}' \in \mathsf{V}' : m_{\mathsf{E}'}(\mathsf{v}') = m\}} \mathbf{1}_B(\mathsf{v}').$$

Furthermore, for $A \in \mathcal{B}((\mathscr{P}_2)^m \times \mathscr{N}_s^{\,m})$,

$$\widehat{\mathbb{Q}}(A) = \frac{1}{\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m)} \cdot \frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)} \mathbb{E} \sum_{(\mathsf{v}', \mathsf{z}'_{(1)\mathsf{v}'} - \mathsf{v}', \ldots, \mathsf{z}'_{(m)\mathsf{v}'} - \mathsf{v}', \Phi_{(1)\mathsf{v}'}, \ldots, \Phi_{(m)\mathsf{v}'})} \mathbf{1}_B(\mathsf{v}') \times$$

$$\times \mathbf{1}\{m_{\mathsf{E}'}(\mathsf{v}') = m\} \mathbf{1}_A(\mathsf{z}'_{(1)\mathsf{v}'} - \mathsf{v}', \ldots, \mathsf{z}'_{(m)\mathsf{v}'} - \mathsf{v}', \Phi_{(1)\mathsf{v}'}, \ldots, \Phi_{(m)\mathsf{v}'}),$$

where $\sum_{(\mathsf{v}', \mathsf{z}'_{(1)\mathsf{v}'} - \mathsf{v}', \ldots, \mathsf{z}'_{(m)\mathsf{v}'} - \mathsf{v}', \Phi_{(1)\mathsf{v}'}, \ldots, \Phi_{(m)\mathsf{v}'})}$ is the sum over all vertices v' of $\mathcal{Y}'$ marked with their m shifted adjacent cells and m corresponding point processes on $\{0\}^2 \times \mathbb{R}$ with intensity 1 if the condition $m_{\mathsf{E}'}(\mathsf{v}') = m$ is fulfilled.

Because each vertex is adjacent to at least 3 cells, using firstly Theorem 1.1.15 for the decomposition of $\widehat{\Theta}$ – the intensity measure of $\widehat{\Phi}$ – and secondly the law of total probability for the decomposition of the mark distribution of $\widehat{\Phi}$, we get

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int_{(\mathscr{P}_2)^m \times \mathscr{N}_s^{\,m}} \int_{\mathbb{R}^2} \sum_{v \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)} + v'} \mathbf{1}_{[0,1]^3}(v)$$

$$\lambda_2(\mathrm{d}v') \widehat{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)})).$$

Write $v := (v_1, v_2, v_3)$. Then

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int_{(\mathscr{P}_2)^m \times \mathscr{N}_s^{\,m}} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)$$

$$\widehat{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)})).$$

The marks $\Phi_{(1)}, \ldots, \Phi_{(m)}$ are i.i.d stationary simple point processes on $\{0\}^2 \times \mathbb{R}$ with intensity 1 and distribution $\mathbb{Q}_1$. Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{V}'}$ the vertex at o has exactly m emanating edges, the joint conditional distribution of $\mathsf{z}'_{(1)}, \ldots, \mathsf{z}'_{(m)}$ is denoted by $\mathbb{Q}'_m$. Since $\Phi_{(1)}, \ldots, \Phi_{(m)}$ are independent of $\mathcal{Y}'$, we infer that

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} \int\limits_{\mathscr{N}_s} \ldots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \bigcup\limits_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)$$

$$\mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \ldots \mathbb{Q}_1(\mathrm{d}\varphi_{(m)}) \mathbb{Q}'_m(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)})).$$

On the other hand, because of Property 2.1.1, we have

$$\int\limits_{\mathscr{N}_s} \ldots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \bigcup\limits_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \ldots \mathbb{Q}_1(\mathrm{d}\varphi_{(m)})$$

$$= \sum_{i=1}^{m} \int\limits_{\mathscr{N}_s} \ldots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \ldots \mathbb{Q}_1(\mathrm{d}\varphi_{(m)})$$

$$= \sum_{i=1}^{m} \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \mathbb{Q}_1(\mathrm{d}\varphi_{(i)}) = \sum_{i=1}^{m} \rho_{z'_{(i)}}.$$

Thus

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} \sum_{i=1}^{m} \rho_{z'_{(i)}} \mathbb{Q}'_m(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)})).$$

Now for fixed $m \geq 3$ and for any planar tessellation T', let $z'_{(1)o}(T'), \ldots, z'_{(m)o}(T')$ be the adjacent cells of the vertex located at the origin o of T' if such a vertex exists and it has exactly m adjacent cells. Otherwise, let $z'_{(1)o}(T') = \ldots = z'_{(m)o}(T') = \emptyset$. We choose $A \in \mathcal{T}'$ as follows

$$A = \{T' \in \mathscr{T}' : (z'_{(1)o}(T'), \ldots, z'_{(m)o}(T')) \in C\},$$

where C is a Borel subset of $(\mathscr{P}_2)^m$. Furthermore, for any vertex v' of $\mathcal{Y}'$ satisfying $m_{\mathsf{Z}'}(\mathsf{v}') = m_{\mathsf{E}'}(\mathsf{v}') = m$, let $\mathsf{z}'_{(1)\mathsf{v}'}, \ldots, \mathsf{z}'_{(m)\mathsf{v}'}$ be its adjacent cells. Define, for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\mathbb{Q}^{(m)}_{\mathsf{V}'}(A) := \frac{1}{\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m)} \cdot \frac{1}{\gamma_{\mathsf{V}'} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{v}' \in \mathsf{V}' : m_{\mathsf{E}'}(\mathsf{v}') = m\}} \mathbf{1}_B(\mathsf{v}') \mathbf{1}_A(\mathcal{Y}' - \mathsf{v}').$$

Then

$$\int\limits_{\mathscr{T}'} \mathbf{1}_C(z'_{(1)o}(T'),\dots,z'_{(m)o}(T'))\mathbb{Q}^{(m)}_{\mathsf{V}'}(\mathrm{d}T') = \int\limits_{\mathscr{T}'} \mathbf{1}_A(T')\mathbb{Q}^{(m)}_{\mathsf{V}'}(\mathrm{d}T') = \mathbb{Q}^{(m)}_{\mathsf{V}'}(A)$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m)}\cdot\frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)}\mathbb{E}\sum_{\{\mathsf{v}'\in\mathsf{V}':m_{\mathsf{E}'}(\mathsf{v}')=m\}} \mathbf{1}_B(\mathsf{v}')\mathbf{1}_A(\mathcal{Y}'-\mathsf{v}')$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m)}\cdot\frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)}\mathbb{E}\sum_{\{\mathsf{v}'\in\mathsf{V}':m_{\mathsf{E}'}(\mathsf{v}')=m\}} \mathbf{1}_B(\mathsf{v}')\times$$

$$\times\, \mathbf{1}_C(z'_{(1)o}(\mathcal{Y}'-\mathsf{v}'),\dots,z'_{(m)o}(\mathcal{Y}'-\mathsf{v}'))$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m)}\cdot\frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)}\mathbb{E}\sum_{(\mathsf{v}',\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\dots,\mathsf{z}'_{(m)\mathsf{v}'}-\mathsf{v}')} \mathbf{1}_B(\mathsf{v}')\mathbf{1}\{m_{\mathsf{E}'}(\mathsf{v}')=m\}\times$$

$$\times\, \mathbf{1}_C(\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\dots,\mathsf{z}'_{(m)\mathsf{v}'}-\mathsf{v}'),$$

where $\sum\limits_{(\mathsf{v}',\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\dots,\mathsf{z}'_{(m)\mathsf{v}'}-\mathsf{v}')}$ is the sum over all vertices v' of $\mathcal{Y}'$ marked with their m shifted adjacent cells if the condition $m_{\mathsf{E}'}(\mathsf{v}')=m$ is fulfilled. We arrive at

$$\int\limits_{\mathscr{T}'} \mathbf{1}_C(z'_{(1)o}(T'),\dots,z'_{(m)o}(T'))\mathbb{Q}^{(m)}_{\mathsf{V}'}(\mathrm{d}T') = \mathbb{Q}'_m(C) =$$

$$= \int\limits_{(\mathscr{P}_2)^m} \mathbf{1}_C(z'_{(1)},\dots,z'_{(m)})\mathbb{Q}'_m(\mathrm{d}(z'_{(1)},\dots,z'_{(m)})).$$

By a standard argument of integration theory, we get

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'}\sum_{m=3}^{\infty}\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m)\int\limits_{(\mathscr{P}_2)^m}\sum_{i=1}^{m}\rho_{z'_{(i)}}\mathbb{Q}'_m(\mathrm{d}(z'_{(1)},\dots,z'_{(m)}))$$

$$= \gamma_{\mathsf{V}'}\sum_{m=3}^{\infty}\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m)\int\limits_{\mathscr{T}'}\sum_{i=1}^{m}\rho_{z'_{(i)o}(T')}\mathbb{Q}^{(m)}_{\mathsf{V}'}(\mathrm{d}T').$$

The latter is $\gamma_{\mathsf{V}'}\int\limits_{\mathscr{T}'}\sum\limits_{\{z'\in Z'(T'):z'\supset\{o\}\}}\rho_{z'}\mathbb{Q}_{\mathsf{V}'}(\mathrm{d}T')$ according to the law of total probability, where $Z'(T')$ is the set of cells of a planar tessellation T'. Finally,

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}\Big(\sum_{\{\mathsf{z}':\mathsf{z}'\supset\mathsf{v}'\}}\rho_{\mathsf{z}'}\Big) = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\alpha_{\mathsf{V}'}) = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'},$$

see Definition 2.1.6 for the last equality.

(v) For each edge e'_j of the stationary random planar tessellation $\mathcal{Y}'$, we denote by z'_{j1} and z'_{j2} its adjacent cells. Moreover, we mark each circumcenter $c(\mathsf{e}'_j)$ with the two shifted adjacent cells $\mathsf{z}'_{j1}-c(\mathsf{e}'_j)$, $\mathsf{z}'_{j2}-c(\mathsf{e}'_j)$ and the corresponding independent point processes Φ_{j1}, Φ_{j2}. We obtain a marked point process, denoted by $\breve{\Phi}$, in the product space $\mathbb{R}^2 \times (\mathscr{P}_2)^2 \times \mathscr{N}_s^2$. Note that the set of horizontal edges of the column tessellation $\mathcal{Y}$, namely, $\mathsf{E}[\mathsf{hor}]$, is uniquely determined by $\breve{\Phi}$.

Denote by $\mathbb{P}_{\mathsf{E}[\mathsf{hor}]}$ the distribution of $\mathsf{E}[\mathsf{hor}]$ and $\mathbb{P}_{\breve{\Phi}}$ the distribution of $\breve{\Phi}$. We have

$$\begin{aligned}
\gamma_{\mathsf{E}[\mathsf{hor}]} &= \int \sum_{e[hor]\in\varkappa} \mathbf{1}_{[0,1]^3}(c(e[hor]))\mathbb{P}_{\mathsf{E}[\mathsf{hor}]}(\mathrm{d}\varkappa)\\
&= \int \sum_{(c(e'_j),z'_{j1}-c(e'_j),z'_{j2}-c(e'_j),\varphi_{j1},\varphi_{j2})\in\breve{\varphi}} \quad \sum_{c\in\frac{1}{\rho_{z'_{j1}-c(e'_j)}}\varphi_{j1}\cup\frac{1}{\rho_{z'_{j2}-c(e'_j)}}\varphi_{j2}+c(e'_j)} \mathbf{1}_{[0,1]^3}(c)\mathbb{P}_{\breve{\Phi}}(\mathrm{d}\breve{\varphi})\\
&= \int_{\mathbb{R}^2\times(\mathscr{P}_2)^2\times\mathscr{N}_s^2} \sum_{c\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}+c'} \mathbf{1}_{[0,1]^3}(c)\breve{\Theta}(\mathrm{d}(c', z'_{(1)}, z'_{(2)}, \varphi_{(1)}, \varphi_{(2)})),
\end{aligned}$$

where $\breve{\Theta}$ is the intensity measure of the marked point process $\breve{\Phi}$. This leads to

$$\gamma_{\mathsf{E}[\mathsf{hor}]} = \gamma_{\mathsf{E}'} \int_{(\mathscr{P}_2)^2\times\mathscr{N}_s^2} \int_{\mathbb{R}^2} \sum_{c\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}+c'} \mathbf{1}_{[0,1]^3}(c)\lambda_2(\mathrm{d}c')\breve{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, z'_{(2)}, \varphi_{(1)}, \varphi_{(2)})).$$

Here $\breve{\mathbb{Q}}$ is the mark distribution of $\breve{\Phi}$, which is the joint distribution of the four marks, denoted by $\mathsf{z}'_{(1)}, \mathsf{z}'_{(2)}, \Phi_{(1)}, \Phi_{(2)}$, of the typical edge-circumcenter in $\mathcal{Y}'$. Write $c := (c_1, c_2, c_3)$. Then

$$\gamma_{\mathsf{E}[\mathsf{hor}]} = \gamma_{\mathsf{E}'} \int_{(\mathscr{P}_2)^2\times\mathscr{N}_s^2} \sum_{(0,0,c_3)\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3)\breve{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, z'_{(2)}, \varphi_{(1)}, \varphi_{(2)})).$$

Let $\mathbb{Q}'_2$ be the joint distribution of $\mathsf{z}'_{(1)}$ and $\mathsf{z}'_{(2)}$. We obtain

$$\begin{aligned}
\gamma_{\mathsf{E}[\mathsf{hor}]} &= \gamma_{\mathsf{E}'} \int_{(\mathscr{P}_2)^2} \int_{\mathscr{N}_s} \int_{\mathscr{N}_s} \sum_{(0,0,c_3)\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3)\\
&\qquad\qquad \mathbb{Q}_1(\mathrm{d}\varphi_{(1)})\mathbb{Q}_1(\mathrm{d}\varphi_{(2)})\mathbb{Q}'_2(\mathrm{d}(z'_{(1)}, z'_{(2)}))\\
&= \gamma_{\mathsf{E}'} \int_{(\mathscr{P}_2)^2} (\rho_{z'_{(1)}} + \rho_{z'_{(2)}})\mathbb{Q}'_2(\mathrm{d}(z'_{(1)}, z'_{(2)})).
\end{aligned}$$

For any planar tessellation T', let $e'_o(T')$ be the edge of T' with circumcenter at the origin o and $z'_{(1)o}(T')$, $z'_{(2)o}(T')$ the adjacent cells of $e'_o(T')$ if such an edge exists. Otherwise, let $e'_o(T') = z'_{(1)o}(T') = z'_{(2)o}(T') = \emptyset$. Moreover, we choose $A \in \mathcal{T}'$ as follows

$$A = \{T' \in \mathscr{T}' : (z'_{(1)o}(T'), z'_{(2)o}(T')) \in C\},$$

where C is a Borel subset of $(\mathscr{P}_2)^2$. Furthermore, for any edge e' of $\mathcal{Y}'$, let $\mathsf{z}'_{(1)\mathsf{e}'}$, $\mathsf{z}'_{(2)\mathsf{e}'}$ be its adjacent cells. Then, for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\begin{aligned}
&\int_{\mathscr{T}'} \mathbf{1}_C(z'_{(1)o}(T'), z'_{(2)o}(T'))\mathbb{Q}_{\mathsf{E}'}(\mathrm{d}T') = \int_{\mathscr{T}'} \mathbf{1}_A(T')\mathbb{Q}_{\mathsf{E}'}(\mathrm{d}T') = \mathbb{Q}_{\mathsf{E}'}(A) \\
&= \frac{1}{\gamma_{\mathsf{E}'}\lambda_2(B)}\mathbb{E}\sum_{\mathsf{e}'\in\mathsf{E}'} \mathbf{1}_B(c(\mathsf{e}'))\mathbf{1}_A(\mathcal{Y}' - c(\mathsf{e}')) \\
&= \frac{1}{\gamma_{\mathsf{E}'}\lambda_2(B)}\mathbb{E}\sum_{\mathsf{e}'\in\mathsf{E}'} \mathbf{1}_B(c(\mathsf{e}'))\mathbf{1}_C(z'_{(1)o}(\mathcal{Y}' - c(\mathsf{e}')), z'_{(2)o}(\mathcal{Y}' - c(\mathsf{e}'))) \\
&= \frac{1}{\gamma_{\mathsf{E}'}\lambda_2(B)}\mathbb{E}\sum_{(c(\mathsf{e}'),\mathsf{z}'_{(1)\mathsf{e}'}-c(\mathsf{e}'),\mathsf{z}'_{(2)\mathsf{e}'}-c(\mathsf{e}'))} \mathbf{1}_B(c(\mathsf{e}'))\mathbf{1}_C(\mathsf{z}'_{(1)\mathsf{e}'} - c(\mathsf{e}'), \mathsf{z}'_{(2)\mathsf{e}'} - c(\mathsf{e}')),
\end{aligned}$$

where $\sum_{(c(\mathsf{e}'),\mathsf{z}'_{(1)\mathsf{e}'}-c(\mathsf{e}'),\mathsf{z}'_{(2)\mathsf{e}'}-c(\mathsf{e}'))}$ is the sum over all edge-circumcenters in $\mathcal{Y}'$ marked with two shifted adjacent cells of the edge. We arrive at

$$\int_{\mathscr{T}'} \mathbf{1}_C(z'_{(1)o}(T'), z'_{(2)o}(T'))\mathbb{Q}_{\mathsf{E}'}(\mathrm{d}T') = \mathbb{Q}'_2(C) = \int_{(\mathscr{P}_2)^2} \mathbf{1}_C(z'_{(1)}, z'_{(2)})\mathbb{Q}'_2(\mathrm{d}(z'_{(1)}, z'_{(2)})).$$

By a standard argument of integration theory, we get

$$\begin{aligned}
\gamma_{\mathsf{E}[\mathsf{hor}]} &= \gamma_{\mathsf{E}'} \int_{(\mathscr{P}_2)^2} (\rho_{z'_{(1)}} + \rho_{z'_{(2)}})\mathbb{Q}'_2(\mathrm{d}(z'_{(1)}, z'_{(2)})) \\
&= \gamma_{\mathsf{E}'} \int_{\mathscr{T}'} (\rho_{z'_{(1)o}(T')} + \rho_{z'_{(2)o}(T')})\mathbb{Q}_{\mathsf{E}'}(\mathrm{d}T') \\
&= \gamma_{\mathsf{E}'} \int_{\mathscr{T}'} \sum_{\{z'\in Z'(T'):z'\supset e'_o(T')\}} \rho_{z'}\mathbb{Q}_{\mathsf{E}'}(\mathrm{d}T') \\
&= \gamma_{\mathsf{E}'}\mathbb{E}_{\mathsf{E}'}(\alpha_{\mathsf{e}'}) = \gamma_{\mathsf{E}'}\overline{\alpha}_{\mathsf{E}'}.
\end{aligned}$$

According to Lemma 2.1.7(i), we have

$$\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(m_{\mathsf{V}'}(\mathsf{z}')\rho_{\mathsf{z}'}) = \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(m_{\mathsf{E}'}(\mathsf{z}')\rho_{\mathsf{z}'}) = \gamma_{\mathsf{E}'}\overline{\alpha}_{\mathsf{E}'}, \tag{23}$$

which yields

$$\gamma_{\mathsf{E}[\mathsf{hor}]} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}.$$

For each vertical edge $e[vert]$ of the column tessellation $\mathcal{Y}$, we choose its reference point $r(e[vert])$ not the circumcenter (or midpoint) but the lower endpoint of this edge which is a vertex of $\mathcal{Y}$. We obtain

$$\gamma_{\mathsf{E}[\mathsf{vert}]} = \int \sum_{e[vert]\in E[vert](y)} \mathbf{1}_{[0,1]^3}(r(e[vert]))\mathbb{P}_{\mathcal{Y}}(\mathrm{d}y) = \int \sum_{v\in V(y)} \mathbf{1}_{[0,1]^3}(v)\mathbb{P}_{\mathcal{Y}}(\mathrm{d}y) = \gamma_{\mathsf{V}},$$

where $E[vert](y)$ and $V(y)$ are the sets of vertical edges and vertices of a realization y of $\mathcal{Y}$ in that order. Because $\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$; see Proposition 2.2.1(ii), we arrive at

$$\gamma_{\mathsf{E[vert]}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}.$$

(iii) Obviously, $\gamma_{\mathsf{E}} = \gamma_{\mathsf{E[hor]}} + \gamma_{\mathsf{E[vert]}} = 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$.

(iv) Following the equation $\gamma_{\mathsf{V}} - \gamma_{\mathsf{E}} + \gamma_{\mathsf{P}} - \gamma_{\mathsf{Z}} = 0$; see [**3**, Section 9.4], we obtain

$$\gamma_{\mathsf{P}} = \gamma_{\mathsf{E}} - \gamma_{\mathsf{V}} + \gamma_{\mathsf{Z}} = 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}.$$

(vi) For each horizontal plate $p[hor]$ of a realization y of $\mathcal{Y}$, its circumcenter $c(p[hor])$ is also the reference point $r(z)$ of the spatial cell z which possesses $p[hor]$ as its base facet. Therefore

$$\gamma_{\mathsf{P[hor]}} = \int \sum_{p[hor]\in P[hor](y)} \mathbf{1}_{[0,1]^3}(c(p[hor]))\mathbb{P}_{\mathcal{Y}}(\mathrm{d}y) = \int \sum_{z\in Z(y)} \mathbf{1}_{[0,1]^3}(r(z))\mathbb{P}_{\mathcal{Y}}(\mathrm{d}y) = \gamma_{\mathsf{Z}},$$

where $P[hor](y)$ is the set of horizontal plates of a realization y of $\mathcal{Y}$. Hence

$$\gamma_{\mathsf{P[hor]}} = \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}.$$

Now $\gamma_{\mathsf{P[vert]}} = \gamma_{\mathsf{P}} - \gamma_{\mathsf{P[hor]}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} - \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$. □

Further intensities of $\mathcal{Y}$ can be calculated using properties of the column tessellation or formulae given in Theorem 2.2.8 and in [**38**]. In the next proposition we present the formulae of some important intensities which will be used later in this chapter. Recall that a plate-side is a 1-dimensional face of a plate.

Proposition 2.2.2. *The intensity γ_{P_1} of plate-sides, the intensity γ_{Z_0} of cell-apices, the intensity γ_{Z_1} of cell-ridges and the intensity γ_{Z_2} of cell-facets of the column tessellation are given as follows*

(i) $\gamma_{\mathsf{P}_1} = \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} + 4\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$,
(ii) $\gamma_{\mathsf{Z}_0} = 2\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}$,
(iii) $\gamma_{\mathsf{Z}_1} = 3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}$,
(iv) $\gamma_{\mathsf{Z}_2} = 2\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}$.

Note that in the calculation of those intensities, the quantity $\overline{\beta}_{\mathsf{V}'}$ is a necessary input.

Before the proof is presented, we generalize the definition of the ownership relation for the case of the column tessellation $\mathcal{Y}$. Denote by $\mathsf{X}_j^{\neq}$, $j < \dim(\mathsf{X}\text{-object})$, the set of all j-dimensional faces of all X-type objects in $\mathcal{Y}$ but without multiplicity. We emphasize that $\mathsf{X}_j^{\neq}$ is not equal to the multiset X_j; see Remarks 1.3.7 and 1.3.8 for the description of the multiset X_j. Furthermore, let $X(y)$ and $X_j^{\neq}(y)$ be the set of X-type objects and the set of all j-dimensional faces of X-type objects of a fixed realization y of $\mathcal{Y}$, respectively.

Definition 2.2.3. An *ownership relation* for y, denoted by $b(y)$, is a subset of $X_j^{\neq}(y) \times X(y)$ defined by $b(y) := \{(x_j, x) \in X_j^{\neq}(y) \times X(y) : x_j \text{ is a } j\text{-face of } x\}$.

Definition 2.2.4. The object x is called an *owner X-type object* of x_j if $(x_j, x) \in b(y)$.

Remark 2.2.5. For the column tessellation $\mathcal{Y}$, $b(\mathcal{Y})$ is a random ownership relation. Without danger of confusion, let we write b instead of $b(\mathcal{Y})$ for brevity.

It is obvious that x is the owner X-type object of $n_j(\mathsf{x})$ its j-dimensional faces (each of these faces belongs to $\mathsf{X}_j^{\neq}$) and that any $\mathsf{x}_j \in \mathsf{X}_j^{\neq}$ can have more than one owner X-type object. For $\mathsf{x}_j \in \mathsf{X}_j^{\neq}$ denote by $n_{\mathsf{X}}(\mathsf{x}_j)$ the number of owner X-type objects of x_j. Note that the notation $n_j(\mathsf{x})$ and $n_{\mathsf{X}}(\mathsf{x}_j)$ have different meanings.

Definition 2.2.6. The multisets X_j, $j < \dim(\mathsf{X}\text{-object})$, is defined as follows

$$\mathsf{X}_j := \{(\mathsf{x}_j, n_{\mathsf{X}}(\mathsf{x}_j)) : \mathsf{x}_j \in \mathsf{X}_j^{\neq}\}.$$

Definition 2.2.7. Given $A \in \mathcal{T}$ and $B \in \mathcal{B}(\mathbb{R}^3)$ with $0 < \lambda_3(B) < \infty$. The probability measure $\mathbb{Q}_{\mathsf{X}_j}$ on $\mathscr{T}$ given by

$$\mathbb{Q}_{\mathsf{X}_j}(A) := \frac{\mathbb{E} \sum\limits_{\mathsf{x}_j \in \mathsf{X}_j^{\neq}} \mathbf{1}_B(c(\mathsf{x}_j)) \mathbf{1}_A(\mathcal{Y} - c(\mathsf{x}_j)) n_{\mathsf{X}}(\mathsf{x}_j)}{\mathbb{E} \sum\limits_{\mathsf{x}_j \in \mathsf{X}_j^{\neq}} \mathbf{1}_B(c(\mathsf{x}_j)) n_{\mathsf{X}}(\mathsf{x}_j)}$$

is called the Palm distribution of the column tessellation $\mathcal{Y}$ with respect to the n_{X}-weighted typical $\mathsf{X}_j^{\neq}$-type object. Furthermore, $\mathbb{Q}_{\mathsf{X}_j}$ can be interpreted as the *Palm distribution of the column tessellation $\mathcal{Y}$ with respect to the typical X_j-type object.*

The probability measure $\mathbb{Q}_{\mathsf{X}_j}$ in Definition 2.2.7 was introduced in [**39**]. Besides, the intensity γ_{X_j} of the multiset X_j can be defined as $\left[\mathbb{E} \sum\limits_{\mathsf{x}_j \in \mathsf{X}_j^{\neq}} \mathbf{1}_B(c(\mathsf{x}_j)) n_{\mathsf{X}}(\mathsf{x}_j)\right] / \lambda_3(B)$. We observe that the multiset X_j has $n_{\mathsf{X}}(\mathsf{x}_j)$ elements x_j.

Proof of Proposition 2.2.2. (i) For $j = 0, 1$ denote by $(\mathsf{P}[\mathsf{hor}])_j$ and $(\mathsf{P}[\mathsf{vert}])_j$ the set of j-dimensional faces of horizontal plates and the multiset of j-dimensional faces of vertical plates in that order. We have

$$\gamma_{\mathsf{P}_1} = \gamma_{(\mathsf{P}[\mathsf{hor}])_1} + \gamma_{(\mathsf{P}[\mathsf{vert}])_1}.$$

Obviously, $\gamma_{(\mathsf{P}[\mathsf{hor}])_1} = \gamma_{(\mathsf{P}[\mathsf{hor}])_0}$. In order to calculate $\gamma_{(\mathsf{P}[\mathsf{hor}])_0}$ – the intensity of 0-faces of horizontal plates in $\mathcal{Y}$, for each vertex v'_j of $\mathcal{Y}'$, we put $\mathsf{N}_j := n_{\mathsf{Z}'}(\mathsf{v}'_j)$ and denote by $\mathsf{z}'_{j1}, \ldots, \mathsf{z}'_{j\mathsf{N}_j}$ the owner cells of v'_j. Moreover, we mark v'_j with its shifted owner cells $\mathsf{z}'_{j1} - \mathsf{v}'_j, \ldots, \mathsf{z}'_{j\mathsf{N}_j} - \mathsf{v}'_j$, the corresponding i.i.d. stationary simple point processes $\Phi_{j1}, \ldots, \Phi_{j\mathsf{N}_j}$ and the random number N_j of its owner cells. We obtain a marked point process, denoted by $\overline{\Phi}$. We emphasize that the marked point process $\overline{\Phi}$ is different from the marked point process $\widehat{\Phi}$ introduced in the proof of Proposition 2.2.1(ii). To generate $\overline{\Phi}$, the last mark of the vertex v'_j is the random number N_j of its owner cells, whereas in the case of $\widehat{\Phi}$, the last mark of v'_j is the random number M_j of its adjacent cells. In particular, if v'_j is a non-π-vertex, then $\mathsf{M}_j = \mathsf{N}_j$, but if v'_j is a π-vertex, then $\mathsf{N}_j = \mathsf{M}_j - 1$.

Note that the set of 0-faces of horizontal plates of the column tessellation $\mathcal{Y}$, namely, $(\mathsf{P}[\mathsf{hor}])_0$, is uniquely determined by $\overline{\Phi}$. For any $\mathsf{v}'_j \in \mathsf{V}'$, the process of 0-faces of horizontal plates of $\mathcal{Y}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'_j}$ is the point process $\bigcup_{i=1}^{\mathsf{N}_j} \frac{1}{\rho_{\mathsf{z}'_{ji}-\mathsf{v}'_j}}\Phi_{ji} + \mathsf{v}'_j$ on $\mathcal{L}_{\mathsf{v}'_j}$. Therefore, if $\mathbb{P}_{\overline{\Phi}}$ denotes the distribution of $\overline{\Phi}$,

$$\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = \int \sum_{(v'_j, z'_{j1}-v'_j,\dots,z'_{jn_j}-v'_j,\varphi_{j1},\dots,\varphi_{jn_j},n_j)\in\overline{\varphi}} \quad \sum_{v\in \bigcup_{i=1}^{n_j} \frac{1}{\rho_{z'_{ji}-v'_j}}\varphi_{ji}+v'_j} \mathbf{1}_{[0,1]^3}(v)\mathbb{P}_{\overline{\Phi}}(\mathrm{d}\overline{\varphi})$$

$$= \int \sum_{v\in \bigcup_{i=1}^{n} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}+v'} \mathbf{1}_{[0,1]^3}(v)\overline{\Theta}(\mathrm{d}(v', z'_{(1)},\dots,z'_{(n)},\varphi_{(1)},\dots,\varphi_{(n)},n)),$$

where $\overline{\Theta}$ is the intensity measure of the marked point process $\overline{\Phi}$. Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{V}'}$ the vertex at the origin o has exactly n owner cells, the joint conditional distribution of the first $2n$ marks $\mathsf{z}'_{(1)},\dots,\mathsf{z}'_{(n)},\Phi_{(1)},\dots,\Phi_{(n)}$ of the vertex at o is denoted by $\overline{\mathbb{Q}}$. By definition for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0<\lambda_2(B)<\infty$,

$$\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n) = \frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)}\mathbb{E}\sum_{\mathsf{v}'\in\mathsf{V}'}\mathbf{1}_B(\mathsf{v}')\mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{v}')=n\}$$

$$= \frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)}\mathbb{E}\sum_{\{\mathsf{v}'\in\mathsf{V}':n_{\mathsf{Z}'}(\mathsf{v}')=n\}}\mathbf{1}_B(\mathsf{v}').$$

Furthermore, for $A\in\mathcal{B}((\mathscr{P}_2)^n\times\mathscr{N}_s^{\,n})$,

$$\overline{\mathbb{Q}}(A) = \frac{1}{\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)}\cdot\frac{1}{\gamma_{\mathsf{V}'}\lambda_2(B)}\mathbb{E}\sum_{(\mathsf{v}',\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\dots,\mathsf{z}'_{(n)\mathsf{v}'}-\mathsf{v}',\Phi_{(1)\mathsf{v}'},\dots,\Phi_{(n)\mathsf{v}'})}\mathbf{1}_B(\mathsf{v}')\times$$

$$\times\mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{v}')=n\}\mathbf{1}_A(\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\dots,\mathsf{z}'_{(n)\mathsf{v}'}-\mathsf{v}',\Phi_{(1)\mathsf{v}'},\dots,\Phi_{(n)\mathsf{v}'}),$$

where $\sum_{(\mathsf{v}',\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\dots,\mathsf{z}'_{(n)\mathsf{v}'}-\mathsf{v}',\Phi_{(1)\mathsf{v}'},\dots,\Phi_{(n)\mathsf{v}'})}$ is the sum over all vertices v' of $\mathcal{Y}'$ marked with their n shifted owner cells and n corresponding point processes on $\{0\}^2\times\mathbb{R}$ with intensity 1 if the condition $n_{\mathsf{Z}'}(\mathsf{v}')=n$ is fulfilled.

From the fact that each vertex has at least 2 owner cells, Theorem 1.1.15 for the decomposition of $\overline{\Theta}$ – the intensity measure of $\overline{\Phi}$ – and the law of total probability for the decomposition of the mark distribution of $\overline{\Phi}$, we obtain

$$\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\int_{(\mathscr{P}_2)^n\times\mathscr{N}_s^n}\int_{\mathbb{R}^2}\sum_{v\in\bigcup_{i=1}^{n}\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}+v'}\mathbf{1}_{[0,1]^3}(v)$$

$$\lambda_2(\mathrm{d}v')\overline{\mathbb{Q}}(\mathrm{d}(z'_{(1)},\dots,z'_{(n)},\varphi_{(1)},\dots,\varphi_{(n)}))$$

$$= \gamma_{\mathsf{V}'} \sum_{n=2}^{\infty} \mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n) \int\limits_{(\mathscr{P}_2)^n \times \mathscr{N}_s^n} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{n} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \overline{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \dots, z'_{(n)}, \varphi_{(1)}, \dots, \varphi_{(n)})).$$

The marks $\Phi_{(1)}, \dots, \Phi_{(n)}$ are i.i.d stationary simple point processes on $\{0\}^2 \times \mathbb{R}$ with intensity 1 and distribution $\mathbb{Q}_1$. Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{V}'}$ the vertex at o has n owner cells, the joint conditional distribution of $\mathsf{z}'_{(1)}, \dots, \mathsf{z}'_{(n)}$ is denoted by $\mathbb{Q}'_n$. Since $\Phi_{(1)}, \dots, \Phi_{(n)}$ are independent of $\mathcal{Y}'$, we infer that

$$\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = \gamma_{\mathsf{V}'} \sum_{n=2}^{\infty} \mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n) \int\limits_{(\mathscr{P}_2)^n} \int\limits_{\mathscr{N}_s} \dots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{n} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \dots \mathbb{Q}_1(\mathrm{d}\varphi_{(n)}) \mathbb{Q}'_n(\mathrm{d}(z'_{(1)}, \dots, z'_{(n)}))$$

$$= \gamma_{\mathsf{V}'} \sum_{n=2}^{\infty} \mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n) \int\limits_{(\mathscr{P}_2)^n} \sum_{i=1}^{n} \rho_{z'_{(i)}} \mathbb{Q}'_n(\mathrm{d}(z'_{(1)}, \dots, z'_{(n)})).$$

For fixed $n \geq 2$ and for any planar tessellation T', let $z'_{(1)o}(T'), \dots, z'_{(n)o}(T')$ be the owner cells of the vertex located at the origin o of T' if such a vertex exists and it has exactly n owner cells. Otherwise, let $z'_{(1)o}(T') = \dots = z'_{(n)o}(T') = \emptyset$. We choose $A = \{T' \in \mathscr{T}' : (z'_{(1)o}(T'), \dots, z'_{(n)o}(T')) \in C\}$, where C is a Borel subset of $(\mathscr{P}_2)^n$. Furthermore, for any vertex v' of $\mathcal{Y}'$ satisfying $n_{\mathsf{Z}'}(\mathsf{v}') = n$, let $\mathsf{z}'_{(1)\mathsf{v}'}, \dots, \mathsf{z}'_{(n)\mathsf{v}'}$ be its owner cells. Define, for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\mathbb{Q}^{(n)}_{\mathsf{V}'}(A) := \frac{1}{\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n)} \cdot \frac{1}{\gamma_{\mathsf{V}'} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{v}' \in \mathsf{V}' : n_{\mathsf{Z}'}(\mathsf{v}') = n\}} \mathbf{1}_B(\mathsf{v}') \mathbf{1}_A(\mathcal{Y}' - \mathsf{v}').$$

Then

$$\int\limits_{\mathscr{T}'} \mathbf{1}_C(z'_{(1)o}(T'), \dots, z'_{(n)o}(T')) \mathbb{Q}^{(n)}_{\mathsf{V}'}(\mathrm{d}T') = \int\limits_{\mathscr{T}'} \mathbf{1}_A(T') \mathbb{Q}^{(n)}_{\mathsf{V}'}(\mathrm{d}T') = \mathbb{Q}^{(n)}_{\mathsf{V}'}(A) =$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n)} \cdot \frac{1}{\gamma_{\mathsf{V}'} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{v}' \in \mathsf{V}' : n_{\mathsf{Z}'}(\mathsf{v}') = n\}} \mathbf{1}_B(\mathsf{v}') \mathbf{1}_A(\mathcal{Y}' - \mathsf{v}')$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n)} \cdot \frac{1}{\gamma_{\mathsf{V}'} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{v}' \in \mathsf{V}' : n_{\mathsf{Z}'}(\mathsf{v}') = n\}} \mathbf{1}_B(\mathsf{v}') \times \times \mathbf{1}_C(z'_{(1)o}(\mathcal{Y}' - \mathsf{v}'), \dots, z'_{(n)o}(\mathcal{Y}' - \mathsf{v}'))$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n)} \cdot \frac{1}{\gamma_{\mathsf{V}'} \lambda_2(B)} \mathbb{E} \sum_{(\mathsf{v}', \mathsf{z}'_{(1)\mathsf{v}'} - \mathsf{v}', \dots, \mathsf{z}'_{(n)\mathsf{v}'} - \mathsf{v}')} \mathbf{1}_B(\mathsf{v}') \mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{v}') = n\} \times \times \mathbf{1}_C(\mathsf{z}'_{(1)\mathsf{v}'} - \mathsf{v}', \dots, \mathsf{z}'_{(n)\mathsf{v}'} - \mathsf{v}'),$$

where $\sum\limits_{(\mathsf{v}',\mathsf{z}'_{(1)\mathsf{v}'}-\mathsf{v}',\ldots,\mathsf{z}'_{(n)\mathsf{v}'}-\mathsf{v}')}$ is the sum over all vertices v' of $\mathcal{Y}'$ marked with their n shifted owner cells if the condition $n_{\mathsf{Z}'}(\mathsf{v}') = n$ is fulfilled. We arrive at

$$\int\limits_{\mathscr{T}'} \mathbf{1}_C(z'_{(1)o}(T'),\ldots,z'_{(n)o}(T'))\mathbb{Q}^{(n)}_{\mathsf{V}'}(\mathrm{d}T') = \mathbb{Q}'_n(C)$$

$$= \int\limits_{(\mathscr{P}_2)^n} \mathbf{1}_C(z'_{(1)},\ldots,z'_{(n)})\mathbb{Q}'_n(\mathrm{d}(z'_{(1)},\ldots,z'_{(n)})).$$

By a standard argument of integration theory, the law of total probability and Definition 2.1.6 in that order, we get

$$\begin{aligned}
\gamma_{(\mathsf{P}[\mathsf{hor}])_0} &= \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\int\limits_{(\mathscr{P}_2)^n}\sum_{i=1}^{n}\rho_{z'_{(i)}}\mathbb{Q}'_n(\mathrm{d}(z'_{(1)},\ldots,z'_{(n)}))\\
&= \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\int\limits_{(\mathscr{P}_2)^n}\sum_{i=1}^{n}\rho_{z'_{(i)o}(T')}\mathbb{Q}^{(n)}_{\mathsf{V}'}(\mathrm{d}T')\\
&= \gamma_{\mathsf{V}'}\int\limits_{\mathscr{T}'}\sum_{\{z'\in Z'(T'):(\{o\},z')\in b(T')\}}\rho_{z'}\mathbb{Q}_{\mathsf{V}'}(\mathrm{d}T')\\
&= \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\beta_{\mathsf{v}'}) = \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.
\end{aligned}$$

Consequently

$$\gamma_{(\mathsf{P}[\mathsf{hor}])_1} = \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

Furthermore, each vertical plate of $\mathcal{Y}$ is a vertical rectangle with 4 sides, two of them are horizontal and two of them are vertical. Hence, using mean value identities in [**39**],

$$\begin{aligned}
\gamma_{(\mathsf{P}[\mathsf{vert}])_1} &= \gamma_{(\mathsf{P}[\mathsf{vert}])_1^{\neq}}\mathbb{E}_{(\mathsf{P}[\mathsf{vert}])_1^{\neq}}(n_{\mathsf{P}[\mathsf{vert}]}((\mathsf{p}[\mathsf{vert}])_1))\\
&= \gamma_{(\mathsf{P}[\mathsf{vert}])_1^{\neq}}\mathbb{E}_{(\mathsf{P}[\mathsf{vert}])_1^{\neq}}\left(\sum_{\mathsf{p}[\mathsf{vert}]\in\mathsf{P}[\mathsf{vert}]:((\mathsf{p}[\mathsf{vert}])_1,\mathsf{p}[\mathsf{vert}])\in b} 1\right)\\
&= \gamma_{\mathsf{P}[\mathsf{vert}]}\mathbb{E}_{\mathsf{P}[\mathsf{vert}]}\left(\sum_{(\mathsf{p}[\mathsf{vert}])_1\in(\mathsf{P}[\mathsf{vert}])_1^{\neq}:((\mathsf{p}[\mathsf{vert}])_1,\mathsf{p}[\mathsf{vert}])\in b} 1\right)\\
&= 4\gamma_{\mathsf{P}[\mathsf{vert}]} = 4\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}.
\end{aligned}$$

Taking the sum of $\gamma_{(\mathsf{P}[\mathsf{hor}])_1}$ and $\gamma_{(\mathsf{P}[\mathsf{vert}])_1}$ gives us the desired statement.

(ii) The set of all cell-apices Z_0 in the column tessellation $\mathcal{Y}$ is a multiset. We have $\mathsf{Z}_0^{\neq} = (\mathsf{P}[\mathsf{hor}])_0$. Note that $\mathsf{Z}_0^{\neq}$ is not equal to the set V of vertices of $\mathcal{Y}$ (for the planar tessellation $\mathcal{Y}'$ we always have $\mathsf{Z}_0'^{\neq} = \mathsf{V}'$). Indeed, if v' is a π-vertex of $\mathcal{Y}'$, denoted by $\mathsf{v}'[\pi]$, then any non-hemi-vertex of $\mathcal{Y}$ appearing on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is not a 0-dimensional face of any horizontal plate. Because each 0-dimensional face $(\mathsf{p}[\mathsf{hor}])_0$

(of some horizontal plate $\mathsf{p}[\mathsf{hor}]$) has 2 owner cells, we obtain

$$\gamma_{\mathsf{Z}_0} = \gamma_{(\mathsf{P}[\mathsf{hor}])_0}\mathbb{E}_{(\mathsf{P}[\mathsf{hor}])_0}(n_{\mathsf{Z}}((\mathsf{p}[\mathsf{hor}])_0)) = 2\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = 2\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

(iii) The column tessellation has only horizontal and vertical cell-ridges. In order to calculate the intensity of the horizontal cell-ridges, we note that $\mathsf{Z}_1[\mathsf{hor}]$ is a multiset and $\mathsf{Z}_1^{\neq}[\mathsf{hor}] = (\mathsf{P}[\mathsf{hor}])_1$. Furthermore, each side $(\mathsf{p}[\mathsf{hor}])_1$ (of some horizontal plate $\mathsf{p}[\mathsf{hor}]$) has 2 owner cells. Consequently,

$$\gamma_{\mathsf{Z}_1[\mathsf{hor}]} = \gamma_{(\mathsf{P}[\mathsf{hor}])_1}\mathbb{E}_{(\mathsf{P}[\mathsf{hor}])_1}(n_{\mathsf{Z}}((\mathsf{p}[\mathsf{hor}])_1)) = 2\gamma_{(\mathsf{P}[\mathsf{hor}])_1} = 2\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

On the other hand, $\mathsf{Z}_1[\mathsf{vert}]$ is a not a multiset. If the reference point of a vertical cell-ridge is its lower endpoint which is a 0-face of a horizontal plate, then obviously

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]} = \gamma_{(\mathsf{P}[\mathsf{hor}])_0} = \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

We obtain

$$\gamma_{\mathsf{Z}_1} = \gamma_{\mathsf{Z}_1[\mathsf{hor}]} + \gamma_{\mathsf{Z}_1[\mathsf{vert}]} = 3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

(iv) $\mathsf{Z}_2[\mathsf{hor}]$ is a multiset and we observe that $\mathsf{Z}_2^{\neq}[\mathsf{hor}] = \mathsf{P}[\mathsf{hor}]$. Moreover, each horizontal plate $\mathsf{p}[\mathsf{hor}]$ has 2 owner cells. Hence

$$\gamma_{\mathsf{Z}_2[\mathsf{hor}]} = \gamma_{\mathsf{P}[\mathsf{hor}]}\mathbb{E}_{\mathsf{P}[\mathsf{hor}]}(n_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}])) = 2\gamma_{\mathsf{P}[\mathsf{hor}]} = 2\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}.$$

$\mathsf{Z}_2[\mathsf{vert}]$ is not a multiset. Each vertical cell-facet of $\mathcal{Y}$ is a vertical rectangle with 4 sides, two of them are horizontal and two of them are vertical. We choose the reference point of a vertical cell-facet as the midpoint of its lower horizontal side which is also a side of a horizontal plate. Hence, obviously

$$\gamma_{\mathsf{Z}_2[\mathsf{vert}]} = \gamma_{(\mathsf{P}[\mathsf{hor}])_1} = \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

The intensity of the cell-facets of the column tessellation $\mathcal{Y}$ is given by

$$\gamma_{\mathsf{Z}_2} = \gamma_{\mathsf{Z}_2[\mathsf{hor}]} + \gamma_{\mathsf{Z}_2[\mathsf{vert}]} = 2\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.$$

□

2.2.2. Formulae for the topological and interior parameters. We now present the three adjacency parameters μ_{VE}, μ_{EP}, μ_{PV} and the four interior parameters ξ, κ, ψ, τ of the column tessellation $\mathcal{Y}$. To clarify their dependence on the basic random planar tessellation $\mathcal{Y}'$, we need from $\mathcal{Y}'$ the mean number of emanating edges of the typical vertex $\mu_{\mathsf{V}'\mathsf{E}'}$, the interior parameter ϕ and five mean values $\overline{\rho}_{\mathsf{Z}'}$, $\overline{\alpha}_{\mathsf{V}'}$, $\overline{\alpha}_{\mathsf{V}'[\pi]}$, $\overline{\beta}_{\mathsf{V}'}$, $\overline{\theta}_{\mathsf{V}'}$ which were already introduced in Definition 2.1.6.

Theorem 2.2.8. *The three topological and four interior parameters of the column tessellation $\mathcal{Y}$ are given by seven parameters of the underlying random planar* *tessellation $\mathcal{Y}'$ and the function $\rho_{\mathsf{Z}'}$ as follows*

$$\mu_{\mathsf{VE}} = 4, \tag{24}$$

$$\mu_{\mathsf{PV}} = \frac{2(3\overline{\alpha}_{\mathsf{V}'} + \overline{\theta}_{\mathsf{V}'})}{2\overline{\alpha}_{\mathsf{V}'} + (\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}, \tag{25}$$

$$\mu_{\mathsf{EP}} = \frac{1}{2}\frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + \frac{3}{2}, \tag{26}$$

$$\xi = \frac{1}{2}\phi\frac{\overline{\alpha}_{\mathsf{V}'[\pi]}}{\overline{\alpha}_{\mathsf{V}'}} + \frac{1}{2}, \tag{27}$$

$$\kappa = \phi\frac{\overline{\alpha}_{\mathsf{V}'[\pi]}}{\overline{\alpha}_{\mathsf{V}'}} + \frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - 1, \tag{28}$$

$$\psi = \frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - \phi\frac{\overline{\alpha}_{\mathsf{V}'[\pi]}}{\overline{\alpha}_{\mathsf{V}'}} - \frac{3\overline{\beta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + 2, \tag{29}$$

$$\tau = \frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - \frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - 1. \tag{30}$$

Note that the Greek letters overset with a bar are derived from $\rho_{\mathsf{Z}'}$ and $\mathcal{Y}'$.

Proof. (24) Each vertex of $\mathcal{Y}$ arises from the intersection of an infinite cylindrical column with a horizontal plane, hence the vertex has 4 outgoing edges; 2 of them are horizontal and the other 2 are vertical and collinear. So we have $m_{\mathsf{E}}(\mathsf{v}) = 4$ for all $\mathsf{v} \in \mathsf{V}$. Therefore $\mu_{\mathsf{VE}} = 4$.

(25) Thanks to Property 2.1.9, we know that for each vertex v' of the planar tessellation $\mathcal{Y}'$, the point process $\Phi_{\mathsf{v}'}$ of vertices of the column tessellation $\mathcal{Y}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'}$ has intensity $\alpha_{\mathsf{v}'}$. Moreover, each point of $\Phi_{\mathsf{v}'}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}') + 3$ plates of $\mathcal{Y}$. Now we mark each vertex v'_j of $\mathcal{Y}'$ with its shifted adjacent cells $\mathsf{z}'_{j1} - \mathsf{v}'_j, \ldots, \mathsf{z}'_{j\mathsf{M}_j} - \mathsf{v}'_j$, the independent point processes $\Phi_{j1}, \ldots, \Phi_{j\mathsf{M}_j}$ and $\mathsf{M}_j = m_{\mathsf{E}'}(\mathsf{v}'_j)$ – the number of emanating edges from v'_j. We obtain again the marked point process $\widehat{\Phi}$. Note that $\widehat{\Phi}$ is already defined in the calculation of the vertex-intensity γ_{V} in Proposition 2.2.1(ii). We find that

$$\begin{aligned}
&\gamma_{\mathsf{V}}\mu_{\mathsf{VP}} = \gamma_{\mathsf{V}}\mathbb{E}_{\mathsf{V}}(m_{\mathsf{P}}(\mathsf{v}))\\
&= \int \sum_{(v'_j, z'_{j1}-v'_j, \ldots, z'_{jm_j}-v'_j, \varphi_{j1}, \ldots, \varphi_{jm_j}, m_j) \in \widehat{\varphi}} \quad \sum_{v \in \bigcup_{i=1}^{m_j} \frac{1}{\rho_{z'_{ji}-v'_j}}\varphi_{ji} + v'_j} \mathbf{1}_{[0,1]^3}(v)(m_j+3)\mathbb{P}_{\widehat{\Phi}}(\mathrm{d}\widehat{\varphi})\\
&= \int (m+3) \sum_{v \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)} + v'} \mathbf{1}_{[0,1]^3}(v)\widehat{\Theta}(\mathrm{d}(v', z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)}, m)),
\end{aligned}$$

which is

$$\gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m \times \mathscr{N}_s^m} (m+3) \int\limits_{\mathbb{R}^2} \sum_{v \in \bigcup\limits_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)} + v'} \mathbf{1}_{[0,1]^3}(v) \lambda_2(\mathrm{d}v')$$
$$\widehat{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)})),$$

recalling the definitions of $\widehat{\Theta}$ and $\widehat{\mathbb{Q}}$ in the proof of Proposition 2.2.1(ii). Write $v := (v_1, v_2, v_3)$. Then, similarly to the calculation of γ_{V}, we get

$$\gamma_{\mathsf{V}} \mu_{\mathsf{VP}} = \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m \times \mathscr{N}_s^m} m \sum_{(0,0,v_3) \in \bigcup\limits_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)$$
$$\widehat{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)})) +$$
$$+ 3\gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m \times \mathscr{N}_s^m} \sum_{(0,0,v_3) \in \bigcup\limits_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}} \varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)$$
$$\widehat{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)}))$$
$$= \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} m \sum_{i=1}^{m} \rho_{z'_{(i)}} \mathbb{Q}'_m(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)})) +$$
$$+ 3\gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} \sum_{i=1}^{m} \rho_{z'_{(i)}} \mathbb{Q}'_m(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}))$$
$$= \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} m \sum_{i=1}^{m} \rho_{z'_{(i)o}(T')} \mathbb{Q}^{(m)}_{\mathsf{V}'}(\mathrm{d}T') +$$
$$+ 3\gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} \sum_{i=1}^{m} \rho_{z'_{(i)o}(T')} \mathbb{Q}^{(m)}_{\mathsf{V}'}(\mathrm{d}T')$$
$$= \gamma_{\mathsf{V}'} \int\limits_{\mathscr{T}'} m_{E'(T')}(o) \sum_{\{z' \in Z'(T') : z' \supset \{o\}\}} \rho_{z'} \mathbb{Q}_{\mathsf{V}'}(\mathrm{d}T') +$$
$$+ 3\gamma_{\mathsf{V}'} \int\limits_{\mathscr{T}'} \sum_{\{z' \in Z'(T') : z' \supset \{o\}\}} \rho_{z'} \mathbb{Q}_{\mathsf{V}'}(\mathrm{d}T')$$
$$= \gamma_{\mathsf{V}'} \mathbb{E}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') \alpha_{\mathsf{v}'}) + 3\gamma_{\mathsf{V}'} \mathbb{E}_{\mathsf{V}'}(\alpha_{\mathsf{v}'}) = \gamma_{\mathsf{V}'} \overline{\theta}_{\mathsf{V}'} + 3\gamma_{\mathsf{V}'} \overline{\alpha}_{\mathsf{V}'},$$

where $E'(T')$ is the set of edges of a planar tessellation T'. With $\gamma_{\mathsf{V}}\mu_{\mathsf{VP}} = \gamma_{\mathsf{P}}\mu_{\mathsf{PV}}$ from Remark 1.3.16 and Proposition 2.2.1(iv) we obtain

$$\mu_{\mathsf{PV}} = \frac{\gamma_{\mathsf{V}}\mu_{\mathsf{VP}}}{\gamma_{\mathsf{P}}} = \frac{\gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} + 3\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}}{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}} = \frac{\overline{\theta}_{\mathsf{V}'} + 3\overline{\alpha}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'} + \frac{1}{2}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}} = \frac{2(3\overline{\alpha}_{\mathsf{V}'} + \overline{\theta}_{\mathsf{V}'})}{2\overline{\alpha}_{\mathsf{V}'} + (\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}.$$

Note that for the third equality, we have used the mean value identity $\gamma_{\mathsf{Z}'} = \frac{1}{2}\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)$ in [**38**, Table 1(a)].

(26) The column tessellation $\mathcal{Y}$ has horizontal and vertical plates. Using Theorem 1.1.11(a), we get, for $B \in \mathcal{B}(\mathbb{R}^3)$ with $0 < \lambda_3(B) < \infty$,

$$\begin{aligned}\gamma_{\mathsf{E}}\mu_{\mathsf{EP}} &= \frac{1}{\lambda_3(B)}\mathbb{E}\sum_{\mathsf{e}\in\mathsf{E}} \mathbf{1}_B(c(\mathsf{e}))m_{\mathsf{P}}(\mathsf{e}) \\ &= \frac{1}{\lambda_3(B)}\mathbb{E}\sum_{\mathsf{e[hor]}\in\mathsf{E[hor]}} \mathbf{1}_B(c(\mathsf{e[hor]}))m_{\mathsf{P}}(\mathsf{e[hor]}) + \\ &\qquad + \frac{1}{\lambda_3(B)}\mathbb{E}\sum_{\mathsf{e[vert]}\in\mathsf{E[vert]}} \mathbf{1}_B(c(\mathsf{e[vert]}))m_{\mathsf{P}}(\mathsf{e[vert]}).\end{aligned}$$

We infer that

$$\gamma_{\mathsf{E}}\mu_{\mathsf{EP}} = \gamma_{\mathsf{E[hor]}}\mu_{\mathsf{E[hor]P}} + \gamma_{\mathsf{E[vert]}}\mu_{\mathsf{E[vert]P}}. \tag{31}$$

We have $\gamma_{\mathsf{E[hor]}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$ from Proposition 2.2.1(v). From the fact that any horizontal edge of $\mathcal{Y}$ is adjacent to 2 vertical plates and 1 horizontal plate (see Property 2.1.10), obviously, $\mu_{\mathsf{E[hor]P}} = 3$. Hence

$$\gamma_{\mathsf{E[hor]}}\mu_{\mathsf{E[hor]P}} = 3\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}.$$

Furthermore, according to Property 2.1.10, each vertical edge of $\mathcal{Y}$ on a vertex-line $\mathcal{L}_{\mathsf{v}'}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}')$ plates. Recalling that the reference point of a vertical edge is its lower endpoint which is a vertex of $\mathcal{Y}$, we get

$$\begin{aligned}&\gamma_{\mathsf{E[vert]}}\mu_{\mathsf{E[vert]P}} \\ &= \gamma_{\mathsf{E[vert]}}\mathbb{E}_{\mathsf{E[vert]}}(m_{\mathsf{P}}(\mathsf{e[vert]})) \\ &= \int \sum_{(v'_j, z'_{j1}-v'_j, \ldots, z'_{jm_j}-v'_j, \varphi_{j1}, \ldots, \varphi_{jm_j}, m_j)\in\widehat{\varphi}} \quad \sum_{v\in\bigcup_{i=1}^{m_j} \frac{1}{\rho_{z'_{ji}-v'_j}}\varphi_{ji}+v'_j} \mathbf{1}_{[0,1]^3}(v)m_j\mathbb{P}_{\widehat{\Phi}}(\mathrm{d}\widehat{\varphi}) \\ &= \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')\alpha_{\mathsf{v}'}) = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\theta_{\mathsf{v}'}) = \gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'}.\end{aligned}$$

The third equality is already shown in the proof of Equation (25). Hence with (iii) of Proposition 2.2.1,

$$\mu_{\mathsf{EP}} = \frac{\gamma_{\mathsf{E[hor]}}\mu_{\mathsf{E[hor]P}} + \gamma_{\mathsf{E[vert]}}\mu_{\mathsf{E[vert]P}}}{\gamma_{\mathsf{E}}} = \frac{3\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'}}{2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}} = \frac{1}{2}\frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + \frac{3}{2}.$$

(27) To find the formulae for the interior parameters we have to consider whether a vertex of $\mathcal{Y}'$ is a π-vertex or not. To calculate the intensity of π-edges $\gamma_{\mathsf{E}[\pi]}$ of $\mathcal{Y}$ we note firstly that all horizontal edges are π-edges and secondly that a vertical

edge is a π-edge if its corresponding vertex v' in $\mathcal{Y}'$ is a π-vertex. Therefore the intensity of the vertical π-edges of $\mathcal{Y}$ is equal to the intensity of the vertices of $\mathcal{Y}$ whose corresponding vertices in $\mathcal{Y}'$ are π-vertices. Similarly to the argument for γ_{V} in Propostion 2.2.1(ii), the latter is $\gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]}$, where $\overline{\alpha}_{\mathsf{V}'[\pi]}$ was introduced in Definition 2.1.6. Combining with $\gamma_{\mathsf{E[hor]}} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$ (see Proposition 2.2.1(v)), we get

$$\gamma_{\mathsf{E}[\pi]} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]},$$

which implies, using $\gamma_{\mathsf{E}[\pi]} = \gamma_{\mathsf{E}}\xi$ from Definition 1.3.13 and $\gamma_{\mathsf{E}} = 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}$ from Proposition 2.2.1(iii), that

$$\xi = \frac{\gamma_{\mathsf{E}[\pi]}}{\gamma_{\mathsf{E}}} = \frac{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]}}{2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}} = \frac{1}{2}\phi\frac{\overline{\alpha}_{\mathsf{V}'[\pi]}}{\overline{\alpha}_{\mathsf{V}'}} + \frac{1}{2}.$$

(28) We consider whether the vertices of the column tessellation $\mathcal{Y}$ are hemi-vertices or not. If the vertex v' of $\mathcal{Y}'$ is not a π-vertex, then $\Phi_{\mathsf{v}'}$ contains only non-hemi-vertices of $\mathcal{Y}$. If the vertex v' is a π-vertex, denoted by $\mathsf{v}'[\pi]$, then the intensity of hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is $\alpha_{\mathsf{v}'[\pi]} - \epsilon_{\mathsf{v}'[\pi]}$; see Property 2.1.9. Hence the intensity of hemi-vertices $\gamma_{\mathsf{V[hemi]}} = \gamma_{\mathsf{V}}\kappa$ is

$$\gamma_{\mathsf{V[hemi]}} = \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}(\alpha_{\mathsf{v}'[\pi]} - \epsilon_{\mathsf{v}'[\pi]}) = \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]} - \gamma_{\mathsf{V}'[\pi]}\overline{\epsilon}_{\mathsf{V}'[\pi]}$$

Using $\gamma_{\mathsf{V}'[\pi]}\overline{\epsilon}_{\mathsf{V}'[\pi]} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}$ from Lemma 2.1.7(iii) together with Proposition 2.2.1(ii), we get

$$\kappa = \frac{\gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]} - \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}} = \phi\frac{\overline{\alpha}_{\mathsf{V}'[\pi]}}{\overline{\alpha}_{\mathsf{V}'}} + \frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - 1.$$

(29) To present the parameter ψ, we have to find out the number of relative ridge-interiors adjacent to a vertex in different cases. If the vertex v' of $\mathcal{Y}'$ is not a π-vertex, denoting v' by $\mathsf{v}'[\overline{\pi}]$, then each point of $\Phi_{\mathsf{v}'[\overline{\pi}]}$ – the point process of vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\overline{\pi}]}$ with intensity $\alpha_{\mathsf{v}'[\overline{\pi}]}$ – is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\overline{\pi}]) - 1$ relative ridge-interiors. If the vertex v' of $\mathcal{Y}'$ is a π-vertex, denoted by $\mathsf{v}'[\pi]$, each of the non-hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi]) + 1$ relative ridge-interiors, and each of the hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 2$ relative ridge-interiors; see Property 2.1.9. Hence

$$\begin{aligned}
\gamma_{\mathsf{V}}\psi &= \gamma_{\mathsf{V}'[\overline{\pi}]}\mathbb{E}_{\mathsf{V}'[\overline{\pi}]}[\alpha_{\mathsf{v}'[\overline{\pi}]}(m_{\mathsf{E}'}(\mathsf{v}'[\overline{\pi}]) - 1)] + \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}[\epsilon_{\mathsf{v}'[\pi]}(m_{\mathsf{E}'}(\mathsf{v}'[\pi]) + 1)] + \\
&\qquad + \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}[(\alpha_{\mathsf{v}'[\pi]} - \epsilon_{\mathsf{v}'[\pi]})(m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 2)] \\
&= \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\alpha_{\mathsf{v}'}m_{\mathsf{E}'}(\mathsf{v}')) - 2\gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\alpha_{\mathsf{v}'}) + \gamma_{\mathsf{V}'[\overline{\pi}]}\mathbb{E}_{\mathsf{V}'[\overline{\pi}]}(\alpha_{\mathsf{v}'[\overline{\pi}]}) + 3\gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}(\epsilon_{\mathsf{v}'[\pi]}) \\
&= \gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} - 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]} + 3\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - 3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} \\
&= \gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} - \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]} - 3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} + 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}.
\end{aligned}$$

Therefore,

$$\psi = \frac{\gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} - \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]} - 3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} + 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}}{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}} = \frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - \phi\frac{\overline{\alpha}_{\mathsf{V}'[\pi]}}{\overline{\alpha}_{\mathsf{V}'}} - \frac{3\overline{\beta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + 2.$$

(30) Now for the last identity we consider how the number of relative plate-side-interiors adjacent to a vertex of $\mathcal{Y}$ depends on the type of corresponding vertex of $\mathcal{Y}'$. If the vertex v' of $\mathcal{Y}'$ is a non-π-vertex, denoted by $\mathsf{v}'[\bar{\pi}]$, then each point of $\Phi_{\mathsf{v}'[\bar{\pi}]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}]) - 2$ relative plate-side-interiors. If v' is a π-vertex, denoted by $\mathsf{v}'[\pi]$, each of the non-hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 1$ relative plate-side-interiors, and each of the hemi-vertices of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 2$ relative plate-side-interiors; see Property 2.1.9. Hence

$$\begin{aligned}
\gamma_{\mathsf{V}}\tau &= \gamma_{\mathsf{V}'[\bar{\pi}]}\mathbb{E}_{\mathsf{V}'[\bar{\pi}]}[\alpha_{\mathsf{v}'[\bar{\pi}]}(m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}]) - 2)] + \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}[\epsilon_{\mathsf{v}'[\pi]}(m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 1)] + \\
&\qquad + \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}[(\alpha_{\mathsf{v}'[\pi]} - \epsilon_{\mathsf{v}'[\pi]})(m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 2)] \\
&= \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\alpha_{\mathsf{v}'} m_{\mathsf{E}'}(\mathsf{v}')) - 2\gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(\alpha_{\mathsf{v}'}) + \gamma_{\mathsf{V}'[\pi]}\mathbb{E}_{\mathsf{V}'[\pi]}(\epsilon_{\mathsf{v}'[\pi]}) \\
&= \gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} - 2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'} \\
&= \gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}.
\end{aligned}$$

Therefore

$$\tau = \frac{\gamma_{\mathsf{V}'}\overline{\theta}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} - \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}} = \frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - \frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} - 1.$$

□

Using identities in [**38**] together with Proposition 2.2.1, Proposition 2.2.2 and Theorem 2.2.8, further mean values can be computed. Some examples are given in the following proposition.

Proposition 2.2.9. *The mean numbers of vertices and edges, respectively, of the typical cell are*

$$\mu_{\mathsf{ZV}} = \frac{2(\overline{\theta}_{\mathsf{V}'} + \overline{\alpha}_{\mathsf{V}'})}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}} \quad \text{and} \quad \mu_{\mathsf{ZE}} = \frac{2(\overline{\theta}_{\mathsf{V}'} + 3\overline{\alpha}_{\mathsf{V}'})}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}},$$

whereas the mean numbers of 0*-faces and* 1*-faces of the typical cell are*

$$\nu_0(\mathsf{Z}) = \frac{4}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)}\frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\rho}_{\mathsf{Z}'}} \quad \text{and} \quad \nu_1(\mathsf{Z}) = \frac{6}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)}\frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\rho}_{\mathsf{Z}'}}.$$

Proof. Using the relation $\mu_{\mathsf{VZ}} = \frac{1}{2}(\mu_{\mathsf{VE}}\mu_{\mathsf{EP}} - 2(\mu_{\mathsf{VE}} - 2))$ and $\mu_{\mathsf{EZ}} = \mu_{\mathsf{EP}}$ in [**38**, Table 2(b)], we obtain

$$\mu_{\mathsf{ZV}} = \frac{\gamma_{\mathsf{V}}}{\gamma_{\mathsf{Z}}}\mu_{\mathsf{VZ}} = \frac{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}}{\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}}\left(\frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + 3 - 2\right) = \frac{2\overline{\alpha}_{\mathsf{V}'}}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}\left(\frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + 1\right) = \frac{2(\overline{\theta}_{\mathsf{V}'} + \overline{\alpha}_{\mathsf{V}'})}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}$$

and

$$\mu_{\mathsf{ZE}} = \frac{\gamma_{\mathsf{E}}}{\gamma_{\mathsf{Z}}}\mu_{\mathsf{EZ}} = \frac{2\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}}{\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}} \cdot \frac{1}{2}\left(\frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + 3\right) = \frac{2\overline{\alpha}_{\mathsf{V}'}}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}\left(\frac{\overline{\theta}_{\mathsf{V}'}}{\overline{\alpha}_{\mathsf{V}'}} + 3\right) = \frac{2(\overline{\theta}_{\mathsf{V}'} + 3\overline{\alpha}_{\mathsf{V}'})}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}.$$

Mean value identities in [**39**] give us $\gamma_{\mathsf{Z}_0} = \gamma_{\mathsf{Z}}\nu_0(\mathsf{Z})$ and $\gamma_{\mathsf{Z}_1} = \gamma_{\mathsf{Z}}\nu_1(\mathsf{Z})$. Thus, for the the mean number of apices and ridges of the typical cell of the column tessellation $\mathcal{Y}$, we find that

$$\nu_0(\mathsf{Z}) = \frac{\gamma_{\mathsf{Z}_0}}{\gamma_{\mathsf{Z}}} = \frac{2\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}{\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}} = \frac{4}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)}\frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\rho}_{\mathsf{Z}'}}, \quad \nu_1(\mathsf{Z}) = \frac{\gamma_{\mathsf{Z}_1}}{\gamma_{\mathsf{Z}}} = \frac{3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}{\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}} = \frac{6}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)}\frac{\overline{\beta}_{\mathsf{V}'}}{\overline{\rho}_{\mathsf{Z}'}}.$$

□

Corollary 2.2.10. *To calculate the intensities and topological/interior parameters of a column tessellation with height 1 from the random planar tessellation, five planar parameters are needed,*

$$\gamma_{\mathsf{V}'}, \quad \mu_{\mathsf{V}'\mathsf{E}'}, \quad \phi, \quad \mu_{\mathsf{E}'\mathsf{V}'[\pi]} \quad \text{and} \quad \mu^{(2)}_{\mathsf{V}'\mathsf{E}'}.$$

The intensities of a column tessellation with height 1 are

$$\gamma_{\mathsf{V}} = \gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}, \quad \gamma_{\mathsf{E}} = 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}, \quad \gamma_{\mathsf{P}} = \frac{1}{2}\gamma_{\mathsf{V}'}(3\mu_{\mathsf{V}'\mathsf{E}'} - 2), \quad \gamma_{\mathsf{Z}} = \frac{1}{2}\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2),$$

the topological parameters are

$$\mu_{\mathsf{VE}} = 4, \quad \mu_{\mathsf{PV}} = \frac{2}{3\mu_{\mathsf{V}'\mathsf{E}'} - 2}(3\mu_{\mathsf{V}'\mathsf{E}'} + \mu^{(2)}_{\mathsf{V}'\mathsf{E}'}), \quad \mu_{\mathsf{EP}} = \frac{1}{2\mu_{\mathsf{V}'\mathsf{E}'}}(3\mu_{\mathsf{V}'\mathsf{E}'} + \mu^{(2)}_{\mathsf{V}'\mathsf{E}'}),$$

and for the interior parameters we obtain

$$\xi = \frac{1}{2} + \frac{\mu_{\mathsf{E}'\mathsf{V}'[\pi]}}{4}, \kappa = \frac{\mu_{\mathsf{E}'\mathsf{V}'[\pi]}}{2} - \frac{\phi}{\mu_{\mathsf{V}'\mathsf{E}'}}, \psi = \frac{\mu^{(2)}_{\mathsf{V}'\mathsf{E}'} + 3\phi}{\mu_{\mathsf{V}'\mathsf{E}'}} - 1 - \frac{\mu_{\mathsf{E}'\mathsf{V}'[\pi]}}{2}, \tau = \frac{\mu^{(2)}_{\mathsf{V}'\mathsf{E}'} + \phi}{\mu_{\mathsf{V}'\mathsf{E}'}} - 2.$$

Proof. Recall that in a column tessellation with height one, $\rho_{\mathsf{z}'} = 1$ for any $\mathsf{z}' \in \mathsf{Z}'$. Hence, Remark 2.1.8 gives us $\overline{\rho}_{\mathsf{Z}'} = 1$ and $\overline{\alpha}_{\mathsf{V}'} = \mu_{\mathsf{V}'\mathsf{E}'}$. Combining with Proposition 2.2.1 and the mean value identity $\gamma_{\mathsf{Z}'} = \frac{1}{2}\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)$ from [**38**, Table 1(a)], we get the formulae for the four intensities. On the other hand, Remark 2.1.8 also gives us $\overline{\alpha}_{\mathsf{V}'[\pi]} = \mu_{\mathsf{V}'[\pi]\mathsf{E}'}$, $\overline{\beta}_{\mathsf{V}'} = \mu_{\mathsf{V}'\mathsf{E}'} - \phi$ and $\overline{\theta}_{\mathsf{V}'} = \mu^{(2)}_{\mathsf{V}'\mathsf{E}'}$. Combining with Theorem 2.2.8 and the mean value identity $\mu_{\mathsf{V}'[\pi]\mathsf{E}'} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}}{2\phi}\mu_{\mathsf{E}'\mathsf{V}'[\pi]}$ from Remark 1.3.16, we can determine the topological and interior parameters. □

Remark 2.2.11. In [**6**] constraints on the topological/interior parameters of random spatial tessellations are considered. The authors showed that the second moment $\mu^{(2)}_{\mathsf{V}'\mathsf{E}'}$ of a random planar tessellation is unbounded. So, in the class of column tessellations with height 1, the mean values μ_{EP}, μ_{PV}, τ and ψ are unbounded. Further constraints follow in Proposition 2.2.12.

Proposition 2.2.12. *The constraints for the topological/interior parameters of the column tessellation $\mathcal{Y}$ with height 1 depending on $\mu_{\mathsf{V}'\mathsf{E}'}$ and ϕ of $\mathcal{Y}'$ are as follows*

$$\frac{36}{7} \leq \frac{2\mu_{\mathsf{V}'\mathsf{E}'}(3 + \mu_{\mathsf{V}'\mathsf{E}'})}{3\mu_{\mathsf{V}'\mathsf{E}'} - 2} \leq \mu_{\mathsf{PV}},$$
$$3 \leq \frac{1}{2}(3 + \mu_{\mathsf{V}'\mathsf{E}'}) \leq \mu_{\mathsf{EP}},$$
$$\frac{1}{2} \leq \frac{1}{2} + \frac{3}{2}\frac{\phi}{\mu_{\mathsf{V}'\mathsf{E}'}} \leq \xi \leq 1 - \frac{3(1-\phi)}{2\mu_{\mathsf{V}'\mathsf{E}'}} \leq 1,$$
$$0 \leq \frac{2\phi}{\mu_{\mathsf{V}'\mathsf{E}'}} \leq \kappa \leq 1 - \frac{3 - 2\phi}{\mu_{\mathsf{V}'\mathsf{E}'}} \leq \frac{3}{4},$$
$$2 \leq \mu_{\mathsf{V}'\mathsf{E}'} + \frac{3}{\mu_{\mathsf{V}'\mathsf{E}'}} - 2 \leq \psi,$$

$$1 \le \mu_{\mathsf{V}'\mathsf{E}'} + \frac{\phi}{\mu_{\mathsf{V}'\mathsf{E}'}} - 2 \le \tau.$$

Proof. For any random planar tessellation we have

$$0 \le \phi \le 1 \quad \text{and} \quad 3 \le \mu_{\mathsf{V}'\mathsf{E}'} \le 6 - 2\phi,$$

as shown in [**38**]. Furthermore it is evident that

$$\mu_{\mathsf{V}'[\pi]\mathsf{E}'} \ge 3 \Leftrightarrow \phi > 0 \text{ and } \mu_{\mathsf{V}'[\bar{\pi}]\mathsf{E}'} \ge 3 \Leftrightarrow \phi < 1.$$

On the other hand, because

$$\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} = \gamma_{\mathsf{V}'[\pi]}\mu_{\mathsf{V}'[\pi]\mathsf{E}'} + \gamma_{\mathsf{V}'[\bar{\pi}]}\mu_{\mathsf{V}'[\bar{\pi}]\mathsf{E}'},$$

it follows that $\mu_{\mathsf{V}'\mathsf{E}'} = \phi\mu_{\mathsf{V}'[\pi]\mathsf{E}'} + (1-\phi)\mu_{\mathsf{V}'[\bar{\pi}]\mathsf{E}'}$ and hence,

$$\phi\mu_{\mathsf{V}'[\pi]\mathsf{E}'} = \mu_{\mathsf{V}'\mathsf{E}'} - (1-\phi)\mu_{\mathsf{V}'[\bar{\pi}]\mathsf{E}'}.$$

Therefore, we obtain the following constraints for the mean number of emanating edges of the typical π-vertex

$$3\phi \le \phi\mu_{\mathsf{V}'[\pi]\mathsf{E}'} \le \mu_{\mathsf{V}'\mathsf{E}'} - 3(1-\phi).$$

Hence the constraints for $\mu_{\mathsf{E}'\mathsf{V}'[\pi]}$ are

$$\frac{6\phi}{\mu_{\mathsf{V}'\mathsf{E}'}} \le \mu_{\mathsf{E}'\mathsf{V}'[\pi]} \le 2 - \frac{6(1-\phi)}{\mu_{\mathsf{V}'\mathsf{E}'}}$$

using $\mu_{\mathsf{E}'\mathsf{V}'[\pi]} = 2\phi\mu_{\mathsf{V}'[\pi]\mathsf{E}'}/\mu_{\mathsf{V}'\mathsf{E}'}$.

Applying these results together with the inequality $\mu^{(2)}_{\mathsf{V}'\mathsf{E}'} \ge (\mu_{\mathsf{V}'\mathsf{E}'})^2$ to Corollary 2.2.10 leads to the constraints for column tessellations with height 1. □

2.2.3. Formulae for the metric mean values.

2.2.3.1. *Formulae involving the lengths of* 1*-dimensional objects.* Firstly we consider mean lengths of 1-dimensional objects for the primitive element sets $\mathsf{X} \in \{\mathsf{E}, \mathsf{P}, \mathsf{Z}\}$ in $\mathcal{Y}$, those denoted by

- $\bar{\ell}_{\mathsf{X}}$ – the mean total length of all 1-faces of the typical X-type object, where dim(X-type object)≥ 1.

This yields, for special object sets,

- $\bar{\ell}_{\mathsf{E}}$ – the mean length of the typical edge,
- $\bar{\ell}_{\mathsf{P}}$ – the mean perimeter of the typical plate,
- $\bar{\ell}_{\mathsf{Z}}$ – the mean total length of all ridges of the typical cell.

We can also define $\bar{\ell}_{\mathsf{X}_k}$ and $\bar{\ell}_{\mathsf{X}[\cdot]}$ in a similar way. For example,

- $\bar{\ell}_{\mathsf{E}[\mathrm{hor}]}$, $\bar{\ell}_{\mathsf{E}[\mathrm{vert}]}$ and $\bar{\ell}_{\mathsf{E}[\pi]}$ – the mean length of the typical horizontal, vertical, and π-edge, respectively,
- $\bar{\ell}_{\mathsf{P}_1}$, $\bar{\ell}_{\mathsf{Z}_1}$ – the mean length of the typical plate-side and the typical ridge, respectively,
- $\bar{\ell}_{\mathsf{Z}_2}$ – the mean perimeter of the typical facet.

The notation above does not include, for instance, the mean total length of all edges of the typical cell. Therefore we use again the adjacency concept, analogous to the mean adjacencies μ_{XY}:

$\bar{\ell}_{\mathsf{XY}}$ – the mean total length of all Y-type objects adjacent to the typical X-type object, where dim(Y-type object)$= 1$.

For $\mathsf{X} = \mathsf{Z}$ and $\mathsf{Y} = \mathsf{E}$ we have

$\bar{\ell}_{\mathsf{ZE}}$ – the mean total length of all edges adjacent to the typical cell.

Some of these $\bar{\ell}_{\mathsf{XY}}$ mean values can be easily determined, for example

$$\bar{\ell}_{\mathsf{PE}} = \bar{\ell}_{\mathsf{P}}, \quad \bar{\ell}_{\mathsf{Z}_1\mathsf{E}} = \bar{\ell}_{\mathsf{Z}_1}, \quad \bar{\ell}_{\mathsf{P}_1\mathsf{E}} = \bar{\ell}_{\mathsf{P}_1},$$

but other examples (see Proposition 2.2.16) are more complicated and demonstrate the necessity of the notation.

Using the additional parameter $\overline{\theta}_{\mathsf{E}'}$ of the planar tessellation $\mathcal{Y}'$ – the length-weighted total ρ-intensity of the cells adjacent to the typical edge; see Definition 2.1.6, we can compute the mean lengths of 1 dimensional objects of the column tessellation.

Theorem 2.2.13. *Three mean total length of all* 1*-faces of the typical primitive elements of the column tessellation are given as follows:*

$$\bar{\ell}_{\mathsf{E}} = \frac{1}{2}\left(\frac{\overline{\theta}_{\mathsf{E}'}}{\overline{\alpha}_{\mathsf{E}'}} + \frac{1}{\overline{\alpha}_{\mathsf{V}'}}\right); \tag{32}$$

$$\bar{\ell}_{\mathsf{P}} = \frac{(3\overline{\theta}_{\mathsf{E}'} + 2)\mu_{\mathsf{V}'\mathsf{E}'}}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'} + \mu_{\mathsf{V}'\mathsf{E}'}\overline{\alpha}_{\mathsf{E}'}}; \tag{33}$$

$$\bar{\ell}_{\mathsf{Z}} = \frac{2(\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}. \tag{34}$$

Proof. (32) Recalling that the column tessellation $\mathcal{Y}$ has only horizontal and vertical edges

$$\gamma_{\mathsf{E}}\bar{\ell}_{\mathsf{E}} = \gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} + \gamma_{\mathsf{E[vert]}}\bar{\ell}_{\mathsf{E[vert]}}.$$

For each edge e'_j with two adjacent cells z'_{j1} and z'_{j2} of the stationary random planar tessellation $\mathcal{Y}'$, we mark its circumcenter $c(\mathsf{e}'_j)$ with $\mathsf{e}'_{jo} := \mathsf{e}'_j - c(\mathsf{e}'_j)$, the two shifted adjacent cells $\mathsf{z}'_{j1} - c(\mathsf{e}'_j), \mathsf{z}'_{j2} - c(\mathsf{e}'_j)$ and the corresponding independent point processes Φ_{j1}, Φ_{j2}. We obtain a marked point process, denoted by $\grave{\Phi}$, in the product space $\mathbb{R}^2 \times \mathscr{P}_1^o \times (\mathscr{P}_2)^2 \times \mathscr{N}_s^2$. Here $\mathscr{P}_1^o$ is the (measurable) space of line segments in $\mathbb{R}^2$ with circumcenter at the origin o. We emphasize that the marked point process $\grave{\Phi}$ is not the marked point process $\breve{\Phi}$ defined in the proof of Proposition 2.2.1(v), because to generate $\grave{\Phi}$, each edge-circumcenter of $\mathcal{Y}$ has one additional mark, namely, the shifted edge with circumcenter at the origin o. Because all the edges of $\Phi_{\mathsf{e}'_j}$ are translations of e'_j (see Subsection 2.1.3), if $\mathbb{P}_{\grave{\Phi}}$ denotes the distribution of $\grave{\Phi}$, we have

$$\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} = \int \sum_{(c(e'_j),e'_{jo},z'_{j1}-c(e'_j),z'_{j2}-c(e'_j),\varphi_{j1},\varphi_{j2})\in\grave{\varphi}} \quad \sum_{c\in\frac{1}{\rho_{z'_{j1}-c(e'_j)}}\varphi_{j1}\cup\frac{1}{\rho_{z'_{j2}-c(e'_j)}}\varphi_{j2}+c(e'_j)} \mathbf{1}_{[0,1]^3}(c)\ell(e'_{jo})\mathbb{P}_{\grave{\Phi}}(\mathrm{d}\grave{\varphi})$$

$$= \int\limits_{\mathbb{R}^2\times\mathscr{P}_1^o\times(\mathscr{P}_2)^2\times\mathscr{N}_s^2} \sum_{c\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}+c'} \mathbf{1}_{[0,1]^3}(c)\ell(e'_o)\grave{\Theta}(\mathrm{d}(c',e'_o,z'_{(1)},z'_{(2)},\varphi_{(1)},\varphi_{(2)})),$$

where $\grave{\Theta}$ is the intensity measure of the marked point process $\grave{\Phi}$. This leads to, using Theorem 1.1.15,

$$\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} = \gamma_{\mathsf{E}'} \int\limits_{\mathscr{P}_1^o\times(\mathscr{P}_2)^2\times\mathscr{N}_s^2} \int\limits_{\mathbb{R}^2} \sum_{c\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}+c'} \mathbf{1}_{[0,1]^3}(c)\ell(e'_o)$$
$$\lambda_2(\mathrm{d}c')\grave{\mathbb{Q}}(\mathrm{d}(e'_o,z'_{(1)},z'_{(2)},\varphi_{(1)},\varphi_{(2)})).$$

Here $\grave{\mathbb{Q}}$ is the mark distribution of $\grave{\Phi}$, which is the joint distribution of the five marks, denoted by $\mathsf{e}'_o,\mathsf{z}'_{(1)},\mathsf{z}'_{(2)},\ \Phi_{(1)},\Phi_{(2)}$, of the typical edge-circumcenter in the random planar tessellation $\mathcal{Y}'$. Write $c:=(c_1,c_2,c_3)$. Then

$$\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}}$$
$$= \gamma_{\mathsf{E}'} \int\limits_{\mathscr{P}_1^o\times(\mathscr{P}_2)^2\times\mathscr{N}_s^2} \sum_{(0,0,c_3)\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3)\ell(e'_o)\grave{\mathbb{Q}}(\mathrm{d}(e'_o,z'_{(1)},z'_{(2)},\varphi_{(1)},\varphi_{(2)})).$$

Let $\mathbb{Q}'_3$ be the joint distribution of e'_o, $\mathsf{z}'_{(1)}$ and $\mathsf{z}'_{(2)}$. Similar to the calculation of $\gamma_{\mathsf{E[hor]}}$ in Proposition 2.2.1(v), we obtain

$$\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} = \gamma_{\mathsf{E}'} \int\limits_{\mathscr{P}_1^o\times(\mathscr{P}_2)^2} \ell(e'_o) \int\limits_{\mathscr{N}_s}\int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3)\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3)$$
$$\mathbb{Q}_1(\mathrm{d}\varphi_{(1)})\mathbb{Q}_1(\mathrm{d}\varphi_{(2)})\mathbb{Q}'_3(\mathrm{d}(e'_o,z'_{(1)},z'_{(2)}))$$
$$= \gamma_{\mathsf{E}'} \int\limits_{\mathscr{P}_1^o\times(\mathscr{P}_2)^2} (\rho_{z'_{(1)}}+\rho_{z'_{(2)}})\ell(e'_o)\mathbb{Q}'_3(\mathrm{d}(e'_o,z'_{(1)},z'_{(2)}))$$
$$= \gamma_{\mathsf{E}'} \int\limits_{\mathscr{T}'} \sum_{\{z'\in Z'(T'):z'\supset e'_o(T')\}} \rho_{z'}\ell(e'_o(T'))\mathbb{Q}_{\mathsf{E}'}(\mathrm{d}T')$$
$$= \gamma_{\mathsf{E}'}\mathbb{E}_{\mathsf{E}'}(\ell(\mathsf{e}')\alpha_{\mathsf{e}'}) = \gamma_{\mathsf{E}'}\mathbb{E}_{\mathsf{E}'}(\theta_{\mathsf{e}'}) = \gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'}.$$

On the other hand, since the reference point of a vertical edge of $\mathcal{Y}$ is its lower endpoint which is a vertex of $\mathcal{Y}$, we have

$$\gamma_{\mathsf{E[vert]}}\bar{\ell}_{\mathsf{E[vert]}}$$
$$= \int \sum_{(v'_j,z'_{j1}-v'_j,\dots,z'_{jm_j}-v'_j,\varphi_{j1},\dots,\varphi_{jm_j},m_j)\in\widehat{\varphi}} \sum_{v\in\bigcup_{i=1}^{m_j}\frac{1}{\rho_{z'_{ji}-v'_j}}\varphi_{ji}+v'_j} \mathbf{1}_{[0,1]^3}(v)\ell(e_v[vert])\mathbb{P}_{\widehat{\Phi}}(\mathrm{d}\widehat{\varphi})$$
$$= \int \sum_{v\in\bigcup_{i=1}^{m}\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}+v'} \mathbf{1}_{[0,1]^3}(v)\ell(e_v[vert])\widehat{\Theta}(\mathrm{d}(v',z'_{(1)},\dots,z'_{(m)},\varphi_{(1)},\dots,\varphi_{(m)},m)),$$

where $\ell(e_v[vert])$ is the distance between $v \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)} + v'$ and the upper consecutive point of v, also belonging to $\bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)} + v'$. Here $\widehat{\Phi}$ is already defined in the proof of Proposition 2.2.1(ii) – the marked point process generated by marking each vertex v'_j of $\mathcal{Y}'$ with its shifted adjacent cells $\mathsf{z}'_{j1} - \mathsf{v}'_j, \ldots, \mathsf{z}'_{j\mathsf{M}_j} - \mathsf{v}'_j$, the corresponding i.i.d. stationary simple point processes $\Phi_{j1}, \ldots, \Phi_{j\mathsf{M}_j}$ and the random number of its emanating edges M_j. Thus

$$\gamma_{\mathsf{E[vert]}}\bar{\ell}_{\mathsf{E[vert]}} = \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m \times \mathscr{N}_s^m} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \times$$

$$\times \ell(e_{(0,0,v_3)}[vert]) \widehat{\mathbb{Q}}(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)}, \varphi_{(1)}, \ldots, \varphi_{(m)}))$$

$$= \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) \int\limits_{(\mathscr{P}_2)^m} \int\limits_{\mathscr{N}_s} \cdots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \ell(e_{(0,0,v_3)}[vert])$$

$$\mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \ldots \mathbb{Q}_1(\mathrm{d}\varphi_{(m)}) \mathbb{Q}'_m(\mathrm{d}(z'_{(1)}, \ldots, z'_{(m)})),$$

Here $\ell(e_{(0,0,v_3)}[vert])$ is the distance between $(0,0,v_3) \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}$ and the upper consecutive point of $(0,0,v_3)$, also belonging to $\bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}$. Recall that $\widehat{\mathbb{Q}}$ ($\mathbb{Q}'_m$, respectively) is the joint conditional distribution of the $2m$ marks (m marks, respectively) $\mathsf{z}'_{(1)}, \ldots, \mathsf{z}'_{(m)}, \Phi_{(1)}, \ldots, \Phi_{(m)}$ ($\mathsf{z}'_{(1)}, \ldots, \mathsf{z}'_{(m)}$, respectively) of the vertex at o given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{V}'}$ the vertex at o has exactly m emanating edges. On the other hand, we have

$$\int\limits_{\mathscr{N}_s} \cdots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \ell(e_{(0,0,v_3)}[vert]) \mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \ldots \mathbb{Q}_1(\mathrm{d}\varphi_{(m)})$$

$$= \int \sum_{z \in Z(T(\mathcal{L}_o))} \mathbf{1}_{[0,1]}(r(z)) \ell(z) \mathbb{P}_{\mathsf{T}(\mathcal{L}_o)}(\mathrm{d}T(\mathcal{L}_o)) = \gamma_{Z(\mathsf{T}(\mathcal{L}_o))}\bar{\ell}_{Z(\mathsf{T}(\mathcal{L}_o))},$$

using Theorem 1.1.11(a) for the last equality. Here $\mathsf{T}(\mathcal{L}_o)$ is the random 1-dimensional tessellation generated by the point process $\bigcup_{i=1}^{m} \frac{1}{\rho_{\mathsf{z}'_{(i)}}}\Phi_{(i)}$ on the vertical line $\mathcal{L}_o$ going through the origin o and $\mathbb{P}_{\mathsf{T}(\mathcal{L}_o)}$ is its distribution. Furthermore, $Z(\mathsf{T}(\mathcal{L}_o))$ and $Z(T(\mathcal{L}_o))$ are the set of cells of $\mathsf{T}(\mathcal{L}_o)$ and the set of cells of a realization $T(\mathcal{L}_o)$ in that order. By [**30**, Equation (10.4)], $\gamma_{Z(\mathsf{T}(\mathcal{L}_o))}\bar{\ell}_{Z(\mathsf{T}(\mathcal{L}_o))} = 1$. Thus

$$\int\limits_{\mathscr{N}_s} \cdots \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \bigcup_{i=1}^{m} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3) \ell(e_{(0,0,v_3)}[vert]) \mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \ldots \mathbb{Q}_1(\mathrm{d}\varphi_{(m)}) = 1 \quad (35)$$

and

$$\gamma_{\mathsf{E[vert]}}\bar{\ell}_{\mathsf{E[vert]}} = \gamma_{\mathsf{V}'}\sum_{m=3}^{\infty}\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m)\int\limits_{(\mathscr{P}_2)^m}\mathbb{Q}'_m(\mathrm{d}(z'_{(1)},\dots,z'_{(m)}))$$
$$= \gamma_{\mathsf{V}'}\sum_{m=3}^{\infty}\mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')=m) = \gamma_{\mathsf{V}'}.$$

With Proposition 2.2.1(iii) and Equation (23), we obtain

$$\bar{\ell}_{\mathsf{E}} = \frac{\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}}+\gamma_{\mathsf{E[vert]}}\bar{\ell}_{\mathsf{E[vert]}}}{\gamma_{\mathsf{E}}} = \frac{\gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'}+\gamma_{\mathsf{V}'}}{2\gamma_{\mathsf{V}'}\bar{\alpha}_{\mathsf{V}'}}$$
$$= \frac{1}{2}\left(\frac{\gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'}}{\gamma_{\mathsf{E}'}\bar{\alpha}_{\mathsf{E}'}}+\frac{\gamma_{\mathsf{V}'}}{\gamma_{\mathsf{V}'}\bar{\alpha}_{\mathsf{V}'}}\right) = \frac{1}{2}\left(\frac{\bar{\theta}_{\mathsf{E}'}}{\bar{\alpha}_{\mathsf{E}'}}+\frac{1}{\bar{\alpha}_{\mathsf{V}'}}\right).$$

(33) Similarly, for the plates of $\mathcal{Y}$ we have

$$\gamma_{\mathsf{P}}\bar{\ell}_{\mathsf{P}} = \gamma_{\mathsf{P[hor]}}\bar{\ell}_{\mathsf{P[hor]}}+\gamma_{\mathsf{P[vert]}}\bar{\ell}_{\mathsf{P[vert]}},$$

where

$$\gamma_{\mathsf{P[hor]}}\bar{\ell}_{\mathsf{P[hor]}} = \int\sum_{(c(z'_j),z'_{jo},\varphi_j)\in\widetilde{\varphi}}\ \sum_{c\in\frac{1}{\rho_{z'_{jo}}}\varphi_j+c(z'_j)}\mathbf{1}_{[0,1]^3}(c)\ell(z'_{jo})\mathbb{P}_{\widetilde{\Phi}}(\mathrm{d}\widetilde{\varphi})$$
$$= \gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\ell(\mathsf{z}')\rho_{\mathsf{z}'}) = \gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'}.$$

We have used Lemma 2.1.7(iv) for the last equality. In order to compute $\gamma_{\mathsf{P[vert]}}\bar{\ell}_{\mathsf{P[vert]}}$, we denote by $(\mathsf{P[vert]})_1$, $(\mathsf{P[vert]})_{1,[\mathsf{hor}]}$ and $(\mathsf{P[vert]})_{1,[\mathsf{vert}]}$ the sets of sides, horizontal sides and vertical sides of vertical plates of $\mathcal{Y}$, respectively. Note that each $(\mathsf{p[vert]})_{1,[\mathsf{vert}]}$ has only 1 owner vertical plate and consequently, $(\mathsf{P[vert]})_{1,[\mathsf{vert}]}$ is not a multiset. Nevertheless, $(\mathsf{P[vert]})_{1,[\mathsf{hor}]}$ is a multiset, $(\mathsf{P[vert]})^{\neq}_{1,[\mathsf{hor}]} = \mathsf{E[hor]}$ and each horizontal edge $\mathsf{e[hor]}$ has 2 owner vertical plates. Hence

$$\gamma_{\mathsf{P[vert]}}\bar{\ell}_{\mathsf{P[vert]}} = \gamma_{\mathsf{P[vert]}}\mathbb{E}_{\mathsf{P[vert]}}(\ell(\mathsf{p[vert]}))$$
$$= \gamma_{\mathsf{P[vert]}}\mathbb{E}_{\mathsf{P[vert]}}\Big(\sum_{(\mathsf{p[vert]})_1\in(\mathsf{P[vert]})^{\neq}_1:((\mathsf{p[vert]})_1,\mathsf{p[vert]})\in b}\ell((\mathsf{p[vert]})_1)\Big)$$
$$= \gamma_{(\mathsf{P[vert]})^{\neq}_1}\mathbb{E}_{(\mathsf{P[vert]})^{\neq}_1}\Big(\ell((\mathsf{p[vert]})_1)\sum_{\mathsf{p[vert]}:((\mathsf{p[vert]})_1,\mathsf{p[vert]})\in b}1\Big)\ \text{(mean value identity in [}\mathbf{39}\text{])}$$
$$= \gamma_{(\mathsf{P[vert]})^{\neq}_{1,[\mathsf{hor}]}}\mathbb{E}_{(\mathsf{P[vert]})^{\neq}_{1,[\mathsf{hor}]}}\Big(\ell((\mathsf{p[vert]})_{1,[\mathsf{hor}]})\sum_{\mathsf{p[vert]}:((\mathsf{p[vert]})_{1,[\mathsf{hor}]},\mathsf{p[vert]})\in b}1\Big)+$$
$$+\gamma_{(\mathsf{P[vert]})_{1,[\mathsf{vert}]}}\mathbb{E}_{(\mathsf{P[vert]})_{1,[\mathsf{vert}]}}\Big(\ell((\mathsf{p[vert]})_{1,[\mathsf{vert}]})\sum_{\mathsf{p[vert]}:((\mathsf{p[vert]})_{1,[\mathsf{vert}]},\mathsf{p[vert]})\in b}1\Big)$$
$$= \gamma_{(\mathsf{P[vert]})_{1,[\mathsf{hor}]}}\bar{\ell}_{(\mathsf{P[vert]})_{1,[\mathsf{hor}]}}+\gamma_{(\mathsf{P[vert]})_{1,[\mathsf{vert}]}}\bar{\ell}_{(\mathsf{P[vert]})_{1,[\mathsf{vert}]}} = \gamma_{(\mathsf{P[vert]})_1}\bar{\ell}_{(\mathsf{P[vert]})_1}$$

where

$$\gamma_{(\mathsf{P[vert]})_{1,[\mathsf{hor}]}}\bar{\ell}_{(\mathsf{P[vert]})_{1,[\mathsf{hor}]}} = \gamma_{\mathsf{E[hor]}}\mathbb{E}_{\mathsf{E[hor]}}(\ell(\mathsf{e[hor]})n_{\mathsf{P[vert]}}(\mathsf{e[hor]})) =$$

$$= \gamma_{\mathsf{E[hor]}} \mathbb{E}_{\mathsf{E[hor]}}(2\ell(\mathsf{e[hor]})) = 2\gamma_{\mathsf{E[hor]}} \bar{\ell}_{\mathsf{E[hor]}} = 2\gamma_{\mathsf{E}'} \overline{\theta}_{\mathsf{E}'};$$

see the proof of Equation (32) for the last equality. On the other hand, since each vertical edge of $\mathcal{Y}$ on $\mathcal{L}_{\mathsf{v}'_j}$ is adjacent to $m_{\mathsf{E}'}(\mathsf{v}'_j)$ vertical sides of $m_{\mathsf{E}'}(\mathsf{v}'_j)$ vertical plates (see Property 2.1.10), similarly to the computation of $\gamma_{\mathsf{E[vert]}} \bar{\ell}_{\mathsf{E[vert]}}$,

$$\gamma_{(\mathsf{P[vert]})_{1,[\mathsf{vert}]}} \bar{\ell}_{(\mathsf{P[vert]})_{1,[\mathsf{vert}]}} =$$

$$= \int \sum_{(v'_j, z'_{j1}-v'_j, \ldots, z'_{jm_j}-v'_j, \varphi_{j1}, \ldots, \varphi_{jm_j}, m_j) \in \widehat{\varphi}} \; \sum_{v \in \frac{1}{\rho_{z'_{ji}-v'_j}} \varphi_{ji} + v'_j} \mathbf{1}_{[0,1]^3}(v) \ell(e_v[vert]) m_j \mathbb{P}_{\widehat{\Phi}}(\mathrm{d}\widehat{\varphi})$$

$$= \gamma_{\mathsf{V}'} \sum_{m=3}^{\infty} m \mathbb{Q}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}') = m) = \gamma_{\mathsf{V}'} \mathbb{E}_{\mathsf{V}'}(m_{\mathsf{E}'}(\mathsf{v}')) = \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'}$$

which implies

$$\gamma_{\mathsf{P[vert]}} \bar{\ell}_{\mathsf{P[vert]}} = 2\gamma_{\mathsf{E}'} \overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'}.$$

We get, using Equation (23),

$$\bar{\ell}_{\mathsf{P}} = \frac{\gamma_{\mathsf{P[hor]}} \bar{\ell}_{\mathsf{P[hor]}} + \gamma_{\mathsf{P[vert]}} \bar{\ell}_{\mathsf{P[vert]}}}{\gamma_{\mathsf{P}}} = \frac{3\gamma_{\mathsf{E}'} \overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'}}{\gamma_{\mathsf{Z}'} \overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{E}'} \overline{\alpha}_{\mathsf{E}'}}$$

$$= \frac{\frac{3}{2} \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'} \overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'}}{\frac{1}{2} \gamma_{\mathsf{V}'} (\mu_{\mathsf{V}'\mathsf{E}'} - 2) \overline{\rho}_{\mathsf{Z}'} + \frac{1}{2} \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'} \overline{\alpha}_{\mathsf{E}'}} = \frac{(3\overline{\theta}_{\mathsf{E}'} + 2) \mu_{\mathsf{V}'\mathsf{E}'}}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2) \overline{\rho}_{\mathsf{Z}'} + \mu_{\mathsf{V}'\mathsf{E}'} \overline{\alpha}_{\mathsf{E}'}}.$$

(34) To determine the mean total length of all ridges of the typical cell we observe that $\mathsf{Z}_2[\mathsf{hor}]$ is a multiset, $\mathsf{Z}_2^{\neq}[\mathsf{hor}] = \mathsf{P}[\mathsf{hor}]$ and for any horizontal plate $\mathsf{p}[\mathsf{hor}]$ we have $n_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}]) = 2$. Furthermore, $n_{\mathsf{Z}}(\mathsf{z}_1[\mathsf{vert}]) = 1$ for any vertical cell-ridge $\mathsf{z}_1[\mathsf{vert}]$, therefore, $\mathsf{Z}_1[\mathsf{vert}]$ is a not a multiset. We get

$$\begin{aligned}
\gamma_{\mathsf{Z}} \bar{\ell}_{\mathsf{Z}} &= \gamma_{\mathsf{Z}} \mathbb{E}_{\mathsf{Z}}(\ell(\mathsf{z})) \\
&= \gamma_{\mathsf{Z}} \mathbb{E}_{\mathsf{Z}} \Big(\sum_{\mathsf{z}_2[\mathsf{hor}] \in \mathsf{Z}_2^{\neq}[\mathsf{hor}] : (\mathsf{z}_2[\mathsf{hor}], \mathsf{z}) \in b} \ell(\mathsf{z}_2[\mathsf{hor}]) + \sum_{\mathsf{z}_1[\mathsf{vert}] \in \mathsf{Z}_1[\mathsf{vert}] : (\mathsf{z}_1[\mathsf{vert}], \mathsf{z}) \in b} \ell(\mathsf{z}_1[\mathsf{vert}]) \Big) \\
&= \gamma_{\mathsf{Z}_2^{\neq}[\mathsf{hor}]} \mathbb{E}_{\mathsf{Z}_2^{\neq}[\mathsf{hor}]} \Big(\ell(\mathsf{z}_2[\mathsf{hor}]) \sum_{\mathsf{z} : (\mathsf{z}_2[\mathsf{hor}], \mathsf{z}) \in b} 1 \Big) + \gamma_{\mathsf{Z}_1[\mathsf{vert}]} \mathbb{E}_{\mathsf{Z}_1[\mathsf{vert}]} \Big(\ell(\mathsf{z}_1[\mathsf{vert}]) \sum_{\mathsf{z} : (\mathsf{z}_1[\mathsf{vert}], \mathsf{z}) \in b} 1 \Big) \\
&= \gamma_{\mathsf{P[hor]}} \mathbb{E}_{\mathsf{P[hor]}} \big(\ell(\mathsf{p}[\mathsf{hor}]) n_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}]) \big) + \gamma_{\mathsf{Z}_1[\mathsf{vert}]} \mathbb{E}_{\mathsf{Z}_1[\mathsf{vert}]} \big(\ell(\mathsf{z}_1[\mathsf{vert}]) n_{\mathsf{Z}}(\mathsf{z}_1[\mathsf{vert}]) \big) \\
&= \gamma_{\mathsf{P[hor]}} \mathbb{E}_{\mathsf{P[hor]}} (2\ell(\mathsf{p}[\mathsf{hor}])) + \gamma_{\mathsf{Z}_1[\mathsf{vert}]} \mathbb{E}_{\mathsf{Z}_1[\mathsf{vert}]} (\ell(\mathsf{z}_1[\mathsf{vert}])) \\
&= 2\gamma_{\mathsf{P[hor]}} \bar{\ell}_{\mathsf{P[hor]}} + \gamma_{\mathsf{Z}_1[\mathsf{vert}]} \bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}.
\end{aligned}$$

We have $\gamma_{\mathsf{P[hor]}} \bar{\ell}_{\mathsf{P[hor]}} = \gamma_{\mathsf{E}'} \overline{\theta}_{\mathsf{E}'}$ from the proof of Equation (33). Recall that the reference point of a vertical cell-ridge in $\mathcal{Y}$ is a 0-face of a horizontal plate. In order to calculate $\gamma_{\mathsf{Z}_1[\mathsf{vert}]} \bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}$, we mark each vertex v'_j of $\mathcal{Y}'$ with its shifted owner cells $\mathsf{z}'_{j1} - \mathsf{v}'_j, \ldots, \mathsf{z}'_{j\mathsf{N}_j} - \mathsf{v}'_j$, the corresponding independent point processes $\Phi_{j1}, \ldots, \Phi_{j\mathsf{N}_j}$

and the random number of its owner cells N_j. We obtain the marked point process $\overline{\Phi}$ introduced in the proof of Proposition 2.2.2(i). We have

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} = \int \sum_{(v'_j, z'_{j1}-v'_j,\ldots,z'_{jn_j}-v'_j,\varphi_{j1},\ldots,\varphi_{jn_j},n_j)\in\bar{\varphi}} \sum_{i=1}^{n_j} \sum_{v\in\frac{1}{\rho_{z'_{ji}-v'_j}}\varphi_{ji}+v'_j} \mathbf{1}_{[0,1]^3}(v)\ell_v \mathbb{P}_{\overline{\Phi}}(\mathrm{d}\bar{\varphi})$$

$$= \int \sum_{i=1}^{n} \sum_{v\in\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}+v'} \mathbf{1}_{[0,1]^3}(v)\ell_v \overline{\Theta}(\mathrm{d}(v', z'_{(1)},\ldots,z'_{(n)},\varphi_{(1)},\ldots,\varphi_{(n)},n)),$$

where for each i between 1 and n and for each $v \in \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)} + v'$, the notation ℓ_v is the distance from v to the upper consecutive point of v, also belonging to $\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)} + v'$. Thus,

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}$$
$$= \gamma_{\mathsf{V}'} \sum_{n=2}^{\infty} \mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n) \int_{(\mathscr{P}_2)^n\times\mathscr{N}_s^n} \sum_{i=1}^{n} \sum_{(0,0,v_3)\in\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)\ell_{(0,0,v_3)}$$
$$\overline{\mathbb{Q}}(\mathrm{d}(z'_{(1)},\ldots,z'_{(n)},\varphi_{(1)},\ldots,\varphi_{(n)}))$$

$$= \gamma_{\mathsf{V}'} \sum_{n=2}^{\infty} \mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}') = n) \sum_{i=1}^{n} \int_{(\mathscr{P}_2)^n} \int_{\mathscr{N}_s} \cdots \int_{\mathscr{N}_s} \sum_{(0,0,v_3)\in\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)\ell_{(0,0,v_3)}$$
$$\mathbb{Q}_1(\mathrm{d}\varphi_{(1)})\ldots\mathbb{Q}_1(\mathrm{d}\varphi_{(n)})\mathbb{Q}'_n(\mathrm{d}(z'_{(1)},\ldots,z'_{(n)})).$$

Here for each i between 1 and n and for each $(0,0,v_3) \in \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}$, the notation $\ell_{(0,0,v_3)}$ is the distance from $(0,0,v_3)$ to the upper consecutive point of $(0,0,v_3)$, also belonging to $\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}$. On the other hand, we have

$$\int_{\mathscr{N}_s} \cdots \int_{\mathscr{N}_s} \sum_{(0,0,v_3)\in\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)\ell_{(0,0,v_3)}\mathbb{Q}_1(\mathrm{d}\varphi_{(1)})\ldots\mathbb{Q}_1(\mathrm{d}\varphi_{(n)})$$
$$= \int_{\mathscr{N}_s} \sum_{(0,0,v_3)\in\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}} \mathbf{1}_{[0,1]}(v_3)\ell_{(0,0,v_3)}\mathbb{Q}_1(\mathrm{d}\varphi_{(i)})$$
$$= \int \sum_{z\in Z(\widehat{T}(\mathcal{L}_o))} \mathbf{1}_{[0,1]}(r(z))\ell(z)\mathbb{P}_{\widehat{\mathsf{T}}(\mathcal{L}_o)}(\mathrm{d}\widehat{T}(\mathcal{L}_o)) = \gamma_{Z(\widehat{\mathsf{T}}(\mathcal{L}_o))}\bar{\ell}_{Z(\widehat{\mathsf{T}}(\mathcal{L}_o))},$$

using Theorem 1.1.11(a) for the last equality. Here $\widehat{\mathsf{T}}(\mathcal{L}_{\mathsf{o}})$ is the random 1-dimensional tessellation generated by the point process $\frac{1}{\rho_{z'_{(i)}}}\Phi_{(i)}$ on the vertical line $\mathcal{L}_o$ going through the origin o and $\mathbb{P}_{\widehat{\mathsf{T}}(\mathcal{L}_{\mathsf{o}})}$ is its distribution. Furthermore, $Z(\widehat{\mathsf{T}}(\mathcal{L}_{\mathsf{o}}))$ and $Z(\widehat{T}(\mathcal{L}_o))$ are the set of cells of $\widehat{\mathsf{T}}(\mathcal{L}_{\mathsf{o}})$ and the set of cells of a realization $\widehat{T}(\mathcal{L}_o)$, respectively. By [**30**, Equation (10.4)], $\gamma_{Z(\widehat{\mathsf{T}}(\mathcal{L}_{\mathsf{o}}))}\bar{\ell}_{Z(\widehat{\mathsf{T}}(\mathcal{L}_{\mathsf{o}}))} = 1$. Consequently

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} = \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\sum_{i=1}^{n}\int\limits_{(\mathscr{P}_2)^n}\mathbb{Q}'_n(\mathrm{d}(z'_{(1)},\dots,z'_{(n)}))$$

$$= \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}n\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n) = \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')) = \gamma_{\mathsf{V}'}\nu_{\mathsf{V}'\mathsf{Z}'}.$$

Combining with Proposition 2.2.1(i) and the mean value identity $\nu_{\mathsf{V}'\mathsf{Z}'} = \mu_{\mathsf{V}'\mathsf{E}'} - \phi$ from Remark 2.1.8, we get

$$\bar{\ell}_{\mathsf{Z}} = \frac{2\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{\ell}_{\mathsf{P}[\mathsf{hor}]} + \gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}}{\gamma_{\mathsf{Z}}} = \frac{2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'}}$$

$$= \frac{\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{\frac{1}{2}\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}} = \frac{2(\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'}}.$$

□

Other mean lengths and mean perimeters of objects in the column tessellation $\mathcal{Y}$ can be computed in the same way by separating the roles of horizontal objects and vertical objects. We present the calculation of some important mean lengths and mean perimeters in the next proposition.

Proposition 2.2.14. *In the column tessellation $\mathcal{Y}$, we have*

(i) *the mean length of the typical π-edge is*

$$\bar{\ell}_{\mathsf{E}[\pi]} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\phi}{2(\overline{\alpha}_{\mathsf{V}'} + \phi\overline{\alpha}_{\mathsf{V}'[\pi]})},$$

(ii) *the mean length of the typical cell-ridge is*

$$\bar{\ell}_{\mathsf{Z}_1} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \mu_{\mathsf{V}'\mathsf{E}'} - \phi}{3\overline{\beta}_{\mathsf{V}'}},$$

(iii) *the mean length of the typical plate-side is*

$$\bar{\ell}_{\mathsf{P}_1} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}(3\overline{\theta}_{\mathsf{E}'} + 2)}{2(\overline{\beta}_{\mathsf{V}'} + 4\overline{\alpha}_{\mathsf{V}'})},$$

(iv) *the mean perimeter of the typical cell-facet is*

$$\bar{\ell}_{\mathsf{Z}_2} = \frac{2(\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'} + \overline{\beta}_{\mathsf{V}'}}.$$

Proof. (i) Recalling that in $\mathcal{Y}$ all horizontal edges are π-edges and a vertical edge is a π-edge if the corresponding vertex $\mathsf{v}' \in \mathcal{Y}'$ is a π-vertex. Hence, similarly to the proof of Equation (32),

$$\gamma_{\mathsf{E}[\pi]}\overline{\ell}_{\mathsf{E}[\pi]} = \gamma_{\mathsf{E}[\mathsf{hor}]}\overline{\ell}_{\mathsf{E}[\mathsf{hor}]} + \gamma_{\mathsf{V}'[\pi]} = \gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\phi.$$

Consequently, using $\gamma_{\mathsf{E}[\pi]} = \gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]}$ shown in the proof of Equation (27), the mean length of the typical π-edge is

$$\overline{\ell}_{\mathsf{E}[\pi]} = \frac{\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\phi}{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'[\pi]}\overline{\alpha}_{\mathsf{V}'[\pi]}} = \frac{\frac{1}{2}\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\phi}{\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'} + \gamma_{\mathsf{V}'}\phi\overline{\alpha}_{\mathsf{V}'[\pi]}} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\phi}{2(\overline{\alpha}_{\mathsf{V}'} + \phi\overline{\alpha}_{\mathsf{V}'[\pi]})}.$$

(ii) The column tessellation $\mathcal{Y}$ has horizontal cell-ridges and vertical cell-ridges

$$\gamma_{\mathsf{Z}_1}\overline{\ell}_{\mathsf{Z}_1} = \gamma_{\mathsf{Z}_1[\mathsf{hor}]}\overline{\ell}_{\mathsf{Z}_1[\mathsf{hor}]} + \gamma_{\mathsf{Z}_1[\mathsf{vert}]}\overline{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}.$$

Note that $\mathsf{Z}_1[\mathsf{hor}]$ is a multiset, $\mathsf{Z}_1^{\neq}[\mathsf{hor}] = (\mathsf{P}[\mathsf{hor}])_1$ und each side of any horizontal plate has 2 owner cells. Moreover, for each $\mathsf{s}'_j \in \mathsf{Z}_1'^{\neq}$, put $\mathsf{K}_j := n_{\mathsf{Z}'}(\mathsf{s}'_j)$ – the random number of owner cells of s'_j – and denote by $\mathsf{z}'_{j1}, \ldots, \mathsf{z}'_{j\mathsf{K}_j}$ the owner cells of s'_j. We mark each circumcenter $c(\mathsf{s}'_j)$ with $\mathsf{s}'_{jo} := \mathsf{s}'_j - c(\mathsf{s}'_j)$, the shifted owner cells $\mathsf{z}'_{j1} - c(\mathsf{s}'_j), \ldots, \mathsf{z}'_{j\mathsf{K}_j} - c(\mathsf{s}'_j)$ of s'_j, the point processes $\Phi_{j1}, \ldots, \Phi_{j\mathsf{K}_j}$ and K_j. We obtain a marked point process, denoted by $\vec{\Phi}$. The distribution of $\vec{\Phi}$ is denoted by $\mathbb{P}_{\vec{\Phi}}$. Note that the set of sides of horizontal plates in $\mathcal{Y}$, namely, $(\mathsf{P}[\mathsf{hor}])_1$, is uniquely determined by $\vec{\Phi}$. Let $\mathcal{L}_{c(\mathsf{s}'_j)}$ be the vertical line going through $c(\mathsf{s}'_j)$. The process of side-circumcenters of horizontal plates of $\mathcal{Y}$ on $\mathcal{L}_{c(\mathsf{s}'_j)}$ is the point process $\bigcup_{i=1}^{\mathsf{K}_j} \frac{1}{\rho_{\mathsf{z}'_{ji}-c(\mathsf{s}'_j)}}\Phi_{ji} + c(\mathsf{s}'_j)$ on $\mathcal{L}_{c(\mathsf{s}'_j)}$. A side of a horizontal plate whose circumcenter belongs to $\bigcup_{i=1}^{\mathsf{K}_j} \frac{1}{\rho_{\mathsf{z}'_{ji}-c(\mathsf{s}'_j)}}\Phi_{ji} + c(\mathsf{s}'_j)$ is a translation of s'_j. Therefore

$$\begin{aligned}
&\gamma_{\mathsf{Z}_1[\mathsf{hor}]}\overline{\ell}_{\mathsf{Z}_1[\mathsf{hor}]} = 2\gamma_{(\mathsf{P}[\mathsf{hor}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{hor}])_1} \\
&= 2\int \sum_{(c(s'_j),s'_{jo},z'_{j1}-c(s'_j),\ldots,z'_{jk_j}-c(s'_j),\varphi_{j1},\ldots,\varphi_{jk_j},k_j)\in\vec{\varphi}} \quad \sum_{c\in\bigcup_{i=1}^{k_j}\frac{1}{\rho_{z'_{ji}-c(s'_j)}}\varphi_{ji}+c(s'_j)} \mathbf{1}_{[0,1]^3}(c)\ell(s'_{jo}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{P}_{\vec{\Phi}}(\mathrm{d}\vec{\varphi}) \\
&= 2\int \sum_{c\in\bigcup_{i=1}^{k}\frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)}+c'} \mathbf{1}_{[0,1]^3}(c)\ell(s'_o)\vec{\Theta}(\mathrm{d}(c', s'_o, z'_{(1)}, \ldots, z'_{(k)}, \varphi_{(1)}, \ldots, \varphi_{(k)}, k)),
\end{aligned}$$

where $\vec{\Theta}$ is the intensity measure of the marked point process $\vec{\Phi}$. Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}$ the side with circumcenter at the origin o has k owner cells, the joint conditional distribution of the first $(2k+1)$ marks

$\mathsf{s}'_o, \mathsf{z}'_{(1)}, \dots, \mathsf{z}'_{(k)}, \Phi_{(1)}, \dots, \Phi_{(k)}$ of the side-circumcenter at o is denoted by $\vec{\mathbb{Q}}$. By definition for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k)$$

$$= \frac{1}{\gamma_{\mathsf{Z}'^{\neq}_1}\lambda_2(B)} \mathbb{E} \sum_{\mathsf{s}' \in \mathsf{Z}'^{\neq}_1} \mathbf{1}_B(c(\mathsf{s}'))\mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{s}') = k\} = \frac{1}{\gamma_{\mathsf{Z}'^{\neq}_1}\lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{s}' \in \mathsf{Z}'^{\neq}_1 : n_{\mathsf{Z}'}(\mathsf{s}') = k\}} \mathbf{1}_B(c(\mathsf{s}')).$$

Furthermore, for $A \in \mathcal{B}(\mathscr{P}^o_1 \times (\mathscr{P}_2)^k \times \mathscr{N}_s^k)$,

$$\vec{\mathbb{Q}}(A) = \frac{1}{\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k)} \cdot \frac{1}{\gamma_{\mathsf{Z}'^{\neq}_1}\lambda_2(B)} \mathbb{E} \sum_{(c(\mathsf{s}'), \mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \dots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}'), \Phi_{(1)\mathsf{s}'}, \dots, \Phi_{(k)\mathsf{s}'})}$$

$$\mathbf{1}_B(c(\mathsf{s}'))\mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{s}') = k\}\mathbf{1}_A(\mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \dots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}'), \Phi_{(1)\mathsf{s}'}, \dots, \Phi_{(k)\mathsf{s}'}),$$

where $\sum_{(c(\mathsf{s}'), \mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \dots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}'), \Phi_{(1)\mathsf{s}'}, \dots, \Phi_{(k)\mathsf{s}'})}$ is the sum over all side-circumcenters $c(\mathsf{s}')$ of $\mathcal{Y}'$ marked with $\mathsf{s}' - c(\mathsf{s}')$, their k shifted owner cells and k corresponding point processes on $\{0\}^2 \times \mathbb{R}$ with intensity 1 if the condition $n_{\mathsf{Z}'}(\mathsf{s}') = k$ is fulfilled. This leads to, using Theorem 1.1.15, the law of total probability together with the fact that each $\mathsf{s}'_j \in \mathsf{Z}'^{\neq}_1$ has 1 or 2 owner cells,

$$\gamma_{\mathsf{Z}_1[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{hor}]} = 2\gamma_{\mathsf{Z}'^{\neq}_1} \sum_{k=1}^{2} \mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k) \int_{\mathscr{P}^o_1 \times (\mathscr{P}_2)^k \times \mathscr{N}_s^k} \int_{\mathbb{R}^2}$$

$$\sum_{c \in \bigcup_{i=1}^{k} \frac{1}{\rho_{z'_{(i)}}}\varphi_{(i)} + c'} \mathbf{1}_{[0,1]^3}(c)\ell(s'_o)\lambda_2(\mathrm{d}c')\vec{\mathbb{Q}}(\mathrm{d}(s'_o, z'_{(1)}, \dots, z'_{(k)}, \varphi_{(1)}, \dots, \varphi_{(k)})).$$

Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}$ the side with circumcenter at the origin o has k owner cells, the joint conditional distribution of $\mathsf{s}'_o, \mathsf{z}'_{(1)}, \dots, \mathsf{z}'_{(k)}$ is denoted by $\mathbb{Q}'_{k+1}$. Thus

$$\gamma_{\mathsf{Z}_1[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{hor}]} = 2\gamma_{\mathsf{Z}'^{\neq}_1} \sum_{k=1}^{2} \mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k) \int_{\mathscr{P}^o_1 \times (\mathscr{P}_2)^k} \sum_{i=1}^{k} \rho_{z'_{(i)}}\ell(s'_o)$$

$$\mathbb{Q}'_{k+1}(\mathrm{d}(s'_o, z'_{(1)}, \dots, z'_{(k)})).$$

Fix $k \in \{1, 2\}$. For any planar tessellation T', let $s'_o(T')$ be the side of T' with circumcenter at the origin o and $z'_{(1)o}(T'), \dots, z'_{(k)o}(T')$ the owner cells of $s'_o(T')$ if such a side exists and it has exactly k owner cells. Otherwise, let $s'_o(T') = z'_{(1)o}(T') = \dots = z'_{(k)o}(T') = \emptyset$. Moreover, we choose

$$A = \{T' \in \mathscr{T}' : (s'_o(T'), z'_{(1)o}(T'), \dots, z'_{(k)o}(T')) \in C\},$$

where C is a Borel subset of $\mathscr{P}_1^o \times (\mathscr{P}_2)^k$. Furthermore, for any side s' of $\mathcal{Y}'$ satisfying $n_{\mathsf{Z}'}(\mathsf{s}') = k$, let $\mathsf{z}'_{(1)\mathsf{s}'}, \ldots, \mathsf{z}'_{(k)\mathsf{s}'}$ be its owner cells. Define, for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\mathbb{Q}^{(k)}_{\mathsf{Z}_1'^{\neq}}(A) := \frac{1}{\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k)} \cdot \frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{s}' \in \mathsf{Z}_1'^{\neq} : n_{\mathsf{Z}'}(\mathsf{s}') = k\}} \mathbf{1}_B(c(\mathsf{s}')) \mathbf{1}_A(\mathcal{Y}' - c(\mathsf{s}')).$$

Then

$$\int_{\mathscr{T}'} \mathbf{1}_C(s_o'(T'), z'_{(1)o}(T'), \ldots, z'_{(k)o}(T')) \mathbb{Q}^{(k)}_{\mathsf{Z}_1'^{\neq}}(\mathrm{d}T') = \int_{\mathscr{T}'} \mathbf{1}_A(T') \mathbb{Q}^{(k)}_{\mathsf{Z}_1'^{\neq}}(\mathrm{d}T') = \mathbb{Q}^{(k)}_{\mathsf{Z}_1'^{\neq}}(A) =$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k)} \cdot \frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{s}' \in \mathsf{Z}_1'^{\neq} : n_{\mathsf{Z}'}(\mathsf{s}') = k\}} \mathbf{1}_B(c(\mathsf{s}')) \mathbf{1}_A(\mathcal{Y}' - c(\mathsf{s}'))$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k)} \cdot \frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}} \lambda_2(B)} \times$$

$$\times \mathbb{E} \sum_{\{\mathsf{s}' \in \mathsf{Z}_1'^{\neq} : n_{\mathsf{Z}'}(\mathsf{s}') = k\}} \mathbf{1}_B(c(\mathsf{s}')) \mathbf{1}_C(s_o'(\mathcal{Y}' - c(\mathsf{s}')), z'_{(1)o}(\mathcal{Y}' - c(\mathsf{s}')), \ldots, z'_{(k)o}(\mathcal{Y}' - c(\mathsf{s}')))$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k)} \cdot \frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}} \lambda_2(B)} \mathbb{E} \sum_{\{\mathsf{s}' \in \mathsf{Z}_1'^{\neq} : n_{\mathsf{Z}'}(\mathsf{s}') = k\}} \mathbf{1}_B(c(\mathsf{s}')) \times$$

$$\times \mathbf{1}_C(\mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \ldots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}'))$$

$$= \frac{1}{\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k)} \cdot \frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}} \lambda_2(B)} \mathbb{E} \sum_{(c(\mathsf{s}'), \mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \ldots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}'))} \mathbf{1}_B(c(\mathsf{s}')) \times$$

$$\times \mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{s}') = k\} \mathbf{1}_C(\mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \ldots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}')),$$

where $\sum_{(c(\mathsf{s}'), \mathsf{s}' - c(\mathsf{s}'), \mathsf{z}'_{(1)\mathsf{s}'} - c(\mathsf{s}'), \ldots, \mathsf{z}'_{(k)\mathsf{s}'} - c(\mathsf{s}'))}$ is the sum over all side-circumcenters $c(\mathsf{s}')$ of $\mathcal{Y}'$ marked with $\mathsf{s}' - c(\mathsf{s}')$ and k shifted owner cells of s' if the condition $n_{\mathsf{Z}'}(\mathsf{s}') = k$ is fulfilled. We arrive at

$$\int_{\mathscr{T}'} \mathbf{1}_C(s_o'(T'), z'_{(1)o}(T'), \ldots, z'_{(k)o}(T')) \mathbb{Q}^{(k)}_{\mathsf{Z}_1'^{\neq}}(\mathrm{d}T') = \mathbb{Q}'_{k+1}(C) =$$

$$= \int_{\mathscr{P}_1^o \times (\mathscr{P}_2)^k} \mathbf{1}_C(s_o', z'_{(1)}, \ldots, z'_{(k)}) \mathbb{Q}'_{k+1}(\mathrm{d}(s_o', z'_{(1)}, \ldots, z'_{(k)})).$$

By a standard argument of integration theory, we get

$$\begin{aligned}
&\gamma_{\mathsf{Z}_1[\mathsf{hor}]}\overline{\ell}_{\mathsf{Z}_1[\mathsf{hor}]}\\
&= 2\gamma_{\mathsf{Z}_1'^{\neq}}\sum_{k=1}^{2}\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}')=k)\int\limits_{\mathscr{P}_1^o\times(\mathscr{P}_2)^k}\sum_{i=1}^{k}\rho_{z'_{(i)}}\ell(s'_o)\mathbb{Q}'_{k+1}(\mathrm{d}(s'_o,z'_{(1)},\dots,z'_{(k)}))\\
&= 2\gamma_{\mathsf{Z}_1'^{\neq}}\sum_{k=1}^{2}\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}')=k)\int\limits_{\mathscr{T}'}\sum_{i=1}^{k}\rho_{z'_{(i)o}(T')}\ell(s'_o(T'))\mathbb{Q}^{(k)}_{\mathsf{Z}_1'^{\neq}}(\mathrm{d}T')\\
&= 2\gamma_{\mathsf{Z}_1'^{\neq}}\int\limits_{\mathscr{T}'}\Bigg(\sum_{\{z'\in Z'(T'):(s'_o(T'),z')\in b(T')\}}\rho_{z'}\Bigg)\ell(s'_o(T'))\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(\mathrm{d}T')\\
&= 2\gamma_{\mathsf{Z}_1'^{\neq}}\mathbb{E}_{\mathsf{Z}_1'^{\neq}}\Bigg(\ell(\mathsf{s}')\sum_{\{\mathsf{z}':(\mathsf{s}',\mathsf{z}')\in b\}}\rho_{\mathsf{z}'}\Bigg) = 2\gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}\Bigg(\rho_{\mathsf{z}'}\sum_{\{\mathsf{s}'\in\mathsf{Z}_1'^{\neq}:(\mathsf{s}',\mathsf{z}')\in b\}}\ell(\mathsf{s}')\Bigg)\\
&= 2\gamma_{\mathsf{Z}'}\mathbb{E}_{\mathsf{Z}'}(\rho_{\mathsf{z}'}\ell(\mathsf{z}')) = 2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}.
\end{aligned}$$

The last equality is already shown in Lemma 2.1.7(iv). On the other hand,

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\overline{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} = \gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi);$$

see the proof of Equation (34). Thus, with $\gamma_{\mathsf{Z}_1} = 3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}$ from Proposition 2.2.2(iii),

$$\overline{\ell}_{\mathsf{Z}_1} = \frac{2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}+\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi)}{3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}} = \frac{\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}+\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi)}{3\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}+\mu_{\mathsf{V}'\mathsf{E}'}-\phi}{3\overline{\beta}_{\mathsf{V}'}}.$$

(iii) Similarly to Equation (31), we have $\gamma_{\mathsf{P}_1}\overline{\ell}_{\mathsf{P}_1} = \gamma_{(\mathsf{P}[\mathsf{hor}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{hor}])_1}+\gamma_{(\mathsf{P}[\mathsf{vert}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{vert}])_1}$. Note that $\gamma_{(\mathsf{P}[\mathsf{hor}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{hor}])_1} = \gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}$, as shown in the calculation of $\gamma_{\mathsf{Z}_1[\mathsf{hor}]}\overline{\ell}_{\mathsf{Z}_1[\mathsf{hor}]}$ in Proposition 2.2.14(ii). Furthermore,

$$\gamma_{(\mathsf{P}[\mathsf{vert}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{vert}])_1} = \gamma_{\mathsf{P}[\mathsf{vert}]}\overline{\ell}_{\mathsf{P}[\mathsf{vert}]} = 2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}+\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'},$$

as established in the proof of Equation (33). Therefore, using Proposition 2.2.2(i) we get

$$\begin{aligned}
\overline{\ell}_{\mathsf{P}_1} &= \frac{\gamma_{(\mathsf{P}[\mathsf{hor}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{hor}])_1}+\gamma_{(\mathsf{P}[\mathsf{vert}])_1}\overline{\ell}_{(\mathsf{P}[\mathsf{vert}])_1}}{\gamma_{\mathsf{P}_1}} = \frac{3\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}+\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}}{\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}+4\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}}\\
&= \frac{\frac{3}{2}\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}+\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}}{\gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}+4\gamma_{\mathsf{V}'}\overline{\alpha}_{\mathsf{V}'}} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}(3\overline{\theta}_{\mathsf{E}'}+2)}{2(\overline{\beta}_{\mathsf{V}'}+4\overline{\alpha}_{\mathsf{V}'})}.
\end{aligned}$$

(iv) Recalling that a column tessellation has only horizontal and vertical cell-facets, thus

$$\gamma_{\mathsf{Z}_2}\overline{\ell}_{\mathsf{Z}_2} = \gamma_{\mathsf{Z}_2[\mathsf{hor}]}\overline{\ell}_{\mathsf{Z}_2[\mathsf{hor}]}+\gamma_{\mathsf{Z}_2[\mathsf{vert}]}\overline{\ell}_{\mathsf{Z}_2[\mathsf{vert}]}.$$

We observe that $\mathsf{Z}_2[\mathsf{hor}]$ is a multiset, $\mathsf{Z}_2^{\neq}[\mathsf{hor}] = \mathsf{P}[\mathsf{hor}]$ and each horizonal plate $\mathsf{p}[\mathsf{hor}]$ has 2 owner cells. Consequently

$$\gamma_{\mathsf{Z}_2[\mathsf{hor}]}\overline{\ell}_{\mathsf{Z}_2[\mathsf{hor}]} = 2\gamma_{\mathsf{P}[\mathsf{hor}]}\overline{\ell}_{\mathsf{P}[\mathsf{hor}]} = 2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}.$$

To calculate $\gamma_{\mathsf{Z}_2[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{vert}]}$, we notice that $(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_{1[\mathsf{hor}]} = (\mathsf{P}[\mathsf{hor}])_1$ and $(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_{1[\mathsf{vert}]} = \mathsf{Z}_1[\mathsf{vert}]$. Moreover, each side of any horizontal plate has 2 owner vertical facets and each vertical ridge has also 2 owner vertical facets. Therefore

$$\begin{aligned}
&\gamma_{\mathsf{Z}_2[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{vert}]}\\
&= \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{vert}]}\big(\ell(\mathsf{z}_2[\mathsf{vert}])\big)\\
&= \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{vert}]}\Big(\sum_{(\mathsf{z}_2[\mathsf{vert}])_1\in(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_1:\ ((\mathsf{z}_2[\mathsf{vert}])_1,\mathsf{z}_2[\mathsf{vert}])\in b} \ell((\mathsf{z}_2[\mathsf{vert}])_1)\Big)\\
&= \gamma_{(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_{1,[\mathsf{hor}]}}\mathbb{E}_{(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_{1,[\mathsf{hor}]}}\Big(\ell((\mathsf{z}_2[\mathsf{vert}])_{1,[\mathsf{hor}]})\sum_{\mathsf{z}_2[\mathsf{vert}]:\ ((\mathsf{z}_2[\mathsf{vert}])_{1,[\mathsf{hor}]},\mathsf{z}_2[\mathsf{vert}])\in b} 1\Big)+\\
&\qquad + \gamma_{(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_{1,[\mathsf{vert}]}}\mathbb{E}_{(\mathsf{Z}_2[\mathsf{vert}])^{\neq}_{1,[\mathsf{vert}]}}\Big(\ell((\mathsf{z}_2[\mathsf{vert}])_{1,[\mathsf{vert}]})\sum_{\mathsf{z}_2[\mathsf{vert}]:\ ((\mathsf{z}_2[\mathsf{vert}])_{1,[\mathsf{vert}]},\mathsf{z}_2[\mathsf{vert}])\in b} 1\Big)\\
&= \gamma_{(\mathsf{P}[\mathsf{hor}])_1}\mathbb{E}_{(\mathsf{P}[\mathsf{hor}])_1}\Big(\ell((\mathsf{p}[\mathsf{hor}])_1)\sum_{\mathsf{z}_2[\mathsf{vert}]:\ ((\mathsf{p}[\mathsf{hor}])_1,\mathsf{z}_2[\mathsf{vert}])\in b} 1\Big)+\\
&\qquad + \gamma_{\mathsf{Z}_1[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_1[\mathsf{vert}]}\Big(\ell(\mathsf{z}_1[\mathsf{vert}])\sum_{\mathsf{z}_2[\mathsf{vert}]:\ (\mathsf{z}_1[\mathsf{vert}],\mathsf{z}_2[\mathsf{vert}])\in b} 1\Big)\\
&= 2\gamma_{(\mathsf{P}[\mathsf{hor}])_1}\bar{\ell}_{(\mathsf{P}[\mathsf{hor}])_1} + 2\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} = 2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi).
\end{aligned}$$

We have used results in the proof of Proposition 2.2.14(ii) for the last equality. Combining with $\gamma_{\mathsf{Z}_2} = 2\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}$ from Proposition 2.2.2(iv), we obtain

$$\begin{aligned}
\bar{\ell}_{\mathsf{Z}_2} &= \frac{\gamma_{\mathsf{Z}_2[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{hor}]} + \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{vert}]}}{\gamma_{\mathsf{Z}_2}} = \frac{4\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{2\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}\\
&= \frac{2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}} = \frac{2(\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'} + \overline{\beta}_{\mathsf{V}'}}.
\end{aligned}$$

□

Remark 2.2.15. Using mean value identities in [**39**], we can compute $\bar{\ell}_{\mathsf{Z}_1}$, $\bar{\ell}_{\mathsf{P}_1}$ and $\bar{\ell}_{\mathsf{Z}_2}$ as follows:

$$\bar{\ell}_{\mathsf{Z}_1} = \frac{\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{Z}}}{\gamma_{\mathsf{Z}_1}}, \quad \bar{\ell}_{\mathsf{P}_1} = \frac{\gamma_{\mathsf{P}}\bar{\ell}_{\mathsf{P}}}{\gamma_{\mathsf{P}_1}}, \quad \bar{\ell}_{\mathsf{Z}_2} = \frac{2\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{Z}}}{\gamma_{\mathsf{Z}_2}},$$

and get the same results.

We also take care of results for some $\bar{\ell}_{\mathsf{XY}}$. It is interesting for us to calculate

$\bar{\ell}_{\mathsf{ZE}}$ – the mean total length of all edges adjacent to the typical cell, and,
$\bar{\ell}_{\mathsf{Z}_2\mathsf{E}}$ – the mean total length of all edges adjacent to the typical facet.

Proposition 2.2.16. *The values of* $\bar{\ell}_{\mathsf{ZE}}$ *and* $\bar{\ell}_{\mathsf{Z_2E}}$ *are given as follows:*

$$\bar{\ell}_{\mathsf{ZE}} = \frac{\mu_{\mathsf{V'E'}}(3\overline{\theta}_{\mathsf{E'}} + 2)}{(\mu_{\mathsf{V'E'}} - 2)\overline{\rho}_{\mathsf{Z'}}}; \tag{36}$$

$$\bar{\ell}_{\mathsf{Z_2E}} = \frac{5\mu_{\mathsf{V'E'}}\overline{\theta}_{\mathsf{E'}} + 4\mu_{\mathsf{V'E'}} - 2\phi}{2[(\mu_{\mathsf{V'E'}} - 2)\overline{\rho}_{\mathsf{Z'}} + \overline{\beta}_{\mathsf{V'}}]}. \tag{37}$$

Proof. (36) A column tessellation has horizontal and vertical edges, so

$$\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE}} = \gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE[hor]}} + \gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE[vert]}}.$$

From the fact that any horizontal edge $\mathsf{e[hor]}$ is adjacent to 3 spatial cells (see Property 2.1.10), using mean value identities in [**39**], we have

$$\begin{aligned}\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE[hor]}} &= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{e[hor]:\ e[hor]\subset z}} \ell(\mathsf{e[hor]})\Big) = \gamma_{\mathsf{E[hor]}}\mathbb{E}_{\mathsf{E[hor]}}\Big(\ell(\mathsf{e[hor]}) \sum_{\mathsf{z:\ z\supset e[hor]}} 1\Big) \\ &= \gamma_{\mathsf{E[hor]}}\mathbb{E}_{\mathsf{E[hor]}}(3\ell(\mathsf{e[hor]})) = 3\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} = 3\gamma_{\mathsf{E'}}\overline{\theta}_{\mathsf{E'}};\end{aligned}$$

see the proof of Equation (32) for the last equality. On the other hand, each vertical edge $\mathsf{e[vert]}$ of $\mathcal{Y}$ appearing on the vertex-line $\mathcal{L}_{\mathsf{v'}}$ through $\mathsf{v'} \in \mathsf{V'}$ is adjacent to $m_{\mathsf{E'}}(\mathsf{v'})$ spatial cells; see Property 2.1.10. Therefore, using mean value identities in [**39**], we get, for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\begin{aligned}&\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE[vert]}} \\ &= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{e[vert]:\ e[vert]\subset z}} \ell(\mathsf{e[vert]})\Big) \\ &= \gamma_{\mathsf{E[vert]}}\mathbb{E}_{\mathsf{E[vert]}}\Big(\ell(\mathsf{e[vert]}) \sum_{\mathsf{z:\ z\supset e[vert]}} 1\Big) \\ &= \int \sum_{(v'_j, z'_{j1}-v'_j,\dots,z'_{jm_j}-v'_j,\varphi_{j1},\dots,\varphi_{jm_j},m_j)\in\widehat{\varphi}} \ \sum_{v\in\bigcup_{i=1}^{m_j}\frac{1}{\rho_{z'_{ji}-v'_j}}\varphi_{ji}+v'_j} \mathbf{1}_{[0,1]^3}(v)\ell(e_v[vert])m_j\mathbb{P}_{\widehat{\Phi}}(\mathrm{d}\widehat{\varphi}) \\ &= \gamma_{\mathsf{(P[vert])_{1[vert]}}}\bar{\ell}_{\mathsf{(P[vert])_{1[vert]}}} = \gamma_{\mathsf{V'}}\mu_{\mathsf{V'E'}};\end{aligned}$$

see the proof of Equation (33) for the last equality. Hence, the mean total length of all edges adjacent to the typical cell is

$$\begin{aligned}\bar{\ell}_{\mathsf{ZE}} &= \frac{\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE[hor]}} + \gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE[vert]}}}{\gamma_{\mathsf{Z}}} = \frac{3\gamma_{\mathsf{E'}}\overline{\theta}_{\mathsf{E'}} + \gamma_{\mathsf{V'}}\mu_{\mathsf{V'E'}}}{\gamma_{\mathsf{Z'}}\overline{\rho}_{\mathsf{Z'}}} \\ &= \frac{\frac{3}{2}\gamma_{\mathsf{V'}}\mu_{\mathsf{V'E'}}\overline{\theta}_{\mathsf{E'}} + \gamma_{\mathsf{V'}}\mu_{\mathsf{V'E'}}}{\frac{1}{2}\gamma_{\mathsf{V'}}(\mu_{\mathsf{V'E'}} - 2)\overline{\rho}_{\mathsf{Z'}}} = \frac{\mu_{\mathsf{V'E'}}(3\overline{\theta}_{\mathsf{E'}} + 2)}{(\mu_{\mathsf{V'E'}} - 2)\overline{\rho}_{\mathsf{Z'}}}.\end{aligned}$$

(37) A column tessellation has horizontal and vertical cell-facets, so

$$\gamma_{\mathsf{Z_2}}\bar{\ell}_{\mathsf{Z_2E}} = \gamma_{\mathsf{Z_2[hor]}}\bar{\ell}_{\mathsf{Z_2[hor]E}} + \gamma_{\mathsf{Z_2[vert]}}\bar{\ell}_{\mathsf{Z_2[vert]E}}.$$

Obviously, $\gamma_{\mathsf{Z}_2[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{hor}]\mathsf{E}} = \gamma_{\mathsf{Z}_2[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{hor}]} = 2\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'}$; see the proof of Proposition 2.2.14(iv). From the fact that $\mathsf{Z}_2[\mathsf{vert}]$ and $\mathsf{Z}_1[\mathsf{vert}]$ are non-multisets, we obtain

$$\begin{aligned}
&\gamma_{\mathsf{Z}_2[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{vert}]\mathsf{E}}\\
&= \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{vert}]}\Big(\sum_{\mathsf{e}:\ \mathsf{e}\subset \mathsf{z}_2[\mathsf{vert}]}\ell(\mathsf{e})\Big)\\
&= \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{vert}]}\bigg(\Big(\sum_{\mathsf{e}:\ \mathsf{e}\subset \mathsf{z}_2[\mathsf{vert}]}\ell(\mathsf{e})\Big)\sum_{\mathsf{z}:\ (\mathsf{z}_2[\mathsf{vert}],\mathsf{z})\in b} 1\bigg)\\
&= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\bigg(\sum_{\mathsf{z}_2[\mathsf{vert}]:\ (\mathsf{z}_2[\mathsf{vert}],\mathsf{z})\in b}\Big(\sum_{\mathsf{e}:\mathsf{e}\subset \mathsf{z}_2[\mathsf{vert}]}\ell(\mathsf{e})\Big)\bigg)\\
&= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{e}:\ \mathsf{e}\subset \mathsf{z}}\ell(\mathsf{e}) + \sum_{\mathsf{z}_1[\mathsf{vert}]:\ (\mathsf{z}_1[\mathsf{vert}],\mathsf{z})\in b}\ell(\mathsf{z}_1[\mathsf{vert}])\Big)\\
&= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{e}:\ \mathsf{e}\subset \mathsf{z}}\ell(\mathsf{e})\Big) + \gamma_{\mathsf{Z}_1[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_1[\mathsf{vert}]}\Big(\ell(\mathsf{z}_1[\mathsf{vert}])\sum_{\mathsf{z}:\ (\mathsf{z}_1[\mathsf{vert}],\mathsf{z})\in b} 1\Big)\\
&= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{e}:\ \mathsf{e}\subset \mathsf{z}}\ell(\mathsf{e})\Big) + \gamma_{\mathsf{Z}_1[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_1[\mathsf{vert}]}\big(\ell(\mathsf{z}_1[\mathsf{vert}])\big)\\
&= \gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE}} + \gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}\\
&= 3\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} + \gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi) = 3\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma_{\mathsf{V}'}\phi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\bar{\ell}_{\mathsf{Z}_2\mathsf{E}} &= \frac{\gamma_{\mathsf{Z}_2[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{hor}]\mathsf{E}} + \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{vert}]\mathsf{E}}}{\gamma_{\mathsf{Z}_2}} = \frac{5\gamma_{\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma_{\mathsf{V}'}\phi}{2\gamma_{\mathsf{Z}'}\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}\\
&= \frac{\frac{5}{2}\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma_{\mathsf{V}'}\phi}{\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\overline{\beta}_{\mathsf{V}'}}\\
&= \frac{5\mu_{\mathsf{V}'\mathsf{E}'}\overline{\theta}_{\mathsf{E}'} + 4\mu_{\mathsf{V}'\mathsf{E}'} - 2\phi}{2[(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\overline{\rho}_{\mathsf{Z}'} + \overline{\beta}_{\mathsf{V}'}]}.
\end{aligned}$$

□

Remark 2.2.17. We can also determine $\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE}}$ and $\gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2\mathsf{E}}$ by using mean value identities in [**39**] as follows:

$$\begin{aligned}
\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE}} &= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{e}:\mathsf{e}\subset\mathsf{z}}\ell(\mathsf{e})\Big) = \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{z}_1:(\mathsf{z}_1,\mathsf{z})\in b}\ \sum_{\mathsf{e}:\mathsf{e}\subset\mathsf{z}_1}\ell(\mathsf{e})\Big) + \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{z}_2:(\mathsf{z}_2,\mathsf{z})\in b}\ \sum_{\mathsf{e}:\mathring{\mathsf{e}}\subset\mathring{\mathsf{z}}_2}\ell(\mathsf{e})\Big)\\
&= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{z}_1:(\mathsf{z}_1,\mathsf{z})\in b}\ell(\mathsf{z}_1)\Big) + \gamma_{\mathsf{Z}_2}\mathbb{E}_{\mathsf{Z}_2}\Big(n_{\mathsf{Z}}(\mathsf{z}_2)\sum_{\mathsf{e}:\mathring{\mathsf{e}}\subset\mathring{\mathsf{z}}_2}\ell(\mathsf{e})\Big)\\
&= \gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{Z}} + \gamma_{\mathsf{E}[\pi]}\mathbb{E}_{\mathsf{E}[\pi]}\Big(\ell(\mathsf{e}[\pi])\sum_{\mathsf{z}_2:\mathring{\mathsf{e}}[\pi]\subset\mathring{\mathsf{z}}_2}n_{\mathsf{Z}}(\mathsf{z}_2)\Big) = \gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{Z}} + \gamma_{\mathsf{E}[\pi]}\bar{\ell}_{\mathsf{E}[\pi]}.
\end{aligned}$$

Here we have used the following two facts: firstly, an edge whose relative interior is contained in the relative interior of a cell-facet must be a π-edge; secondly, such a cell-facet is uniquely determined and has only 1 owner cell. Similarly

$$\begin{aligned}\gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2\mathsf{E}} &= \gamma_{\mathsf{Z}_2}\mathbb{E}_{\mathsf{Z}_2}\Big(\sum_{\mathsf{e}:\mathsf{e}\subset\mathsf{z}_2}\ell(\mathsf{e})\Big) = \gamma_{\mathsf{Z}_2}\mathbb{E}_{\mathsf{Z}_2}\Big(\sum_{\mathsf{e}:\mathsf{e}\subset\partial\mathsf{z}_2}\ell(\mathsf{e})\Big) + \gamma_{\mathsf{Z}_2}\mathbb{E}_{\mathsf{Z}_2}\Big(\sum_{\mathsf{e}:\mathring{\mathsf{e}}\subset\mathring{\mathsf{z}}_2}\ell(\mathsf{e})\Big)\\ &= \gamma_{\mathsf{Z}_2}\mathbb{E}_{\mathsf{Z}_2}(\ell(\mathsf{z}_2)) + \gamma_{\mathsf{E}[\pi]}\mathbb{E}_{\mathsf{E}[\pi]}\Big(\sum_{\mathsf{z}_2:\mathring{\mathsf{e}}[\pi]\subset\mathring{\mathsf{z}}_2}\ell(\mathsf{e}[\pi])\Big) = \gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2} + \gamma_{\mathsf{E}[\pi]}\bar{\ell}_{\mathsf{E}[\pi]},\end{aligned}$$

where ∂p denotes the boundary of a polytope p in $\mathbb{R}^d$. The formulae for $\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{Z}}$, $\gamma_{\mathsf{E}[\pi]}\bar{\ell}_{\mathsf{E}[\pi]}$ and $\gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2}$ shown in the proofs of Equation (34), Proposition 2.2.14(i),(iv) lead to

$$\gamma_{\mathsf{Z}}\bar{\ell}_{\mathsf{ZE}} = 2\gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi) + \gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\phi = 3\gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}$$

and

$$\gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2\mathsf{E}} = 4\gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi) + \gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}\phi = 5\gamma_{\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma_{\mathsf{V}'}\phi.$$

Remark 2.2.18. The definition of $\bar{\ell}_{\mathsf{XY}}$ can be extended to the case dim(Y-type object)= 2. In this case, $\bar{\ell}_{\mathsf{XY}}$ is the mean total perimeter of all Y-type objects adjacent to the typical X-type object. We can further compute some other $\bar{\ell}_{\mathsf{XY}}$ mean values, for example

$\bar{\ell}_{\mathsf{VE}}$	–	the mean total length of all edges adjacent to the typical vertex,
$\bar{\ell}_{\mathsf{ZP}}$	–	the mean total perimeter of all plates adjacent to the typical cell,
$\bar{\ell}_{\mathsf{Z}_2\mathsf{P}}$	–	the mean total perimeter of all plates adjacent to the typical facet.

From the fact that each vertex of any tessellation is adjacent to 2 edges, using mean value identities in [**39**] as well as Proposition 2.2.1(ii),(iii) and Equation (32), we obtain

$$\bar{\ell}_{\mathsf{VE}} = \frac{2\gamma_{\mathsf{E}}\bar{\ell}_{\mathsf{E}}}{\gamma_{\mathsf{V}}} = \frac{4\gamma_{\mathsf{V}'}\bar{\alpha}_{\mathsf{V}'}\bar{\ell}_{\mathsf{E}}}{\gamma_{\mathsf{V}'}\bar{\alpha}_{\mathsf{V}'}} = 4\bar{\ell}_{\mathsf{E}} = 2\left(\frac{\bar{\theta}_{\mathsf{E}'}}{\bar{\alpha}_{\mathsf{E}'}} + \frac{1}{\bar{\alpha}_{\mathsf{V}'}}\right).$$

From the fact that each plate of a spatial tessellation is adjacent to 2 cells and 2 cell-facets, using mean value identities in [**39**] together with Proposition 2.2.1(i), Proposition 2.2.2(iv) and Equation (33), we get

$$\bar{\ell}_{\mathsf{ZP}} = \frac{2\gamma_{\mathsf{P}}\bar{\ell}_{\mathsf{P}}}{\gamma_{\mathsf{Z}}} = \frac{3\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}}{\frac{1}{2}\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\bar{\rho}_{\mathsf{Z}'}} = \frac{2(3\bar{\theta}_{\mathsf{E}'} + 2)\mu_{\mathsf{V}'\mathsf{E}'}}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\bar{\rho}_{\mathsf{Z}'}},$$

$$\bar{\ell}_{\mathsf{Z}_2\mathsf{P}} = \frac{2\gamma_{\mathsf{P}}\bar{\ell}_{\mathsf{P}}}{\gamma_{\mathsf{Z}_2}} = \frac{3\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\bar{\theta}_{\mathsf{E}'} + 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}}{\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\bar{\rho}_{\mathsf{Z}'} + \gamma_{\mathsf{V}'}\bar{\beta}_{\mathsf{V}'}} = \frac{(3\bar{\theta}_{\mathsf{E}'} + 2)\mu_{\mathsf{V}'\mathsf{E}'}}{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\bar{\rho}_{\mathsf{Z}'} + \bar{\beta}_{\mathsf{V}'}}.$$

Corollary 2.2.19. *For the mean lengths of* 1*-dimensional objects in the column tessellation of constant cell-height* 1*, we have* (*using the metric parameter* $\bar{\ell}_{\mathsf{E}'}$ *from the random planar tessellation*)

$$\bar{\ell}_{\mathsf{E}} = \frac{1}{2}\left(\bar{\ell}_{\mathsf{E}'} + \frac{1}{\mu_{\mathsf{V}'\mathsf{E}'}}\right), \quad \bar{\ell}_{\mathsf{P}} = \frac{2\mu_{\mathsf{V}'\mathsf{E}'}(3\bar{\ell}_{\mathsf{E}'} + 1)}{3\mu_{\mathsf{V}'\mathsf{E}'} - 2}, \quad \bar{\ell}_{\mathsf{Z}} = \frac{2(2\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} + \mu_{\mathsf{V}'\mathsf{E}'} - \phi)}{\mu_{\mathsf{V}'\mathsf{E}'} - 2},$$

$$\bar{\ell}_{\mathsf{E}[\pi]} = \frac{2(\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} + \phi)}{\mu_{\mathsf{V}'\mathsf{E}'}(2 + \mu_{\mathsf{E}'\mathsf{V}'[\pi]})}, \quad \bar{\ell}_{\mathsf{Z}_1} = \frac{1}{3} + \frac{2\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{3(\mu_{\mathsf{V}'\mathsf{E}'} - \phi)}, \quad \bar{\ell}_{\mathsf{P}_1} = \frac{\mu_{\mathsf{V}'\mathsf{E}'}(3\bar{\ell}_{\mathsf{E}'} + 1)}{5\mu_{\mathsf{V}'\mathsf{E}'} - \phi},$$

$$\bar{\ell}_{\mathsf{Z}_2} = \frac{2(2\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + \mu_{\mathsf{V'E'}} - \phi)}{2\mu_{\mathsf{V'E'}} - \phi - 2}, \bar{\ell}_{\mathsf{ZE}} = \frac{2\mu_{\mathsf{V'E'}}(3\bar{\ell}_{\mathsf{E'}} + 1)}{\mu_{\mathsf{V'E'}} - 2}, \bar{\ell}_{\mathsf{Z_2E}} = \frac{5\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + 2\mu_{\mathsf{V'E'}} - \phi}{2\mu_{\mathsf{V'E'}} - \phi - 2},$$

$$\bar{\ell}_{\mathsf{VE}} = 2\left(\bar{\ell}_{\mathsf{E'}} + \frac{1}{\mu_{\mathsf{V'E'}}}\right), \quad \bar{\ell}_{\mathsf{ZP}} = \frac{4\mu_{\mathsf{V'E'}}(3\bar{\ell}_{\mathsf{E'}} + 1)}{\mu_{\mathsf{V'E'}} - 2}, \quad \bar{\ell}_{\mathsf{Z_2P}} = \frac{2\mu_{\mathsf{V'E'}}(3\bar{\ell}_{\mathsf{E'}} + 1)}{2\mu_{\mathsf{V'E'}} - \phi - 2}.$$

Proof. Remark 2.1.8 gives us the desired statement. □

2.2.3.2. *Formulae for the areas and volumes.* First we deal with areas, using analogous notation. Later in this sub-subsection, we consider volumes.

$\bar{a}_{\mathsf{X}}$ – the mean total area of all 2-faces of the typical X-type object, where dim(X-type object)≥ 2.

In this case we have

- $\bar{a}_{\mathsf{P}}$ – the mean area of the typical plate,
- $\bar{a}_{\mathsf{Z}}$ – the mean surface area of the typical cell, and,
- $\bar{a}_{\mathsf{Z}_2}$ – the mean area of the typical cell-facet.

To determine mean values of the forms $\bar{a}_{\mathsf{X}}$, $\bar{a}_{\mathsf{X_k}}$ and $\bar{a}_{\mathsf{X}[\cdot]}$ of the column tessellation $\mathcal{Y}$, we use two additional mean values of the random planar tessellation $\mathcal{Y}'$, namely $\bar{\theta}_{\mathsf{Z'}}$ and $\bar{\ell}_{\mathsf{E'}}$.

Theorem 2.2.20. *The mean area of the typical plate of the column tessellation is*

$$\bar{a}_{\mathsf{P}} = \frac{(\mu_{\mathsf{V'E'}} - 2)\bar{\theta}_{\mathsf{Z'}} + \mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}}}{(\mu_{\mathsf{V'E'}} - 2)\bar{\rho}_{\mathsf{Z'}} + \mu_{\mathsf{V'E'}}\bar{\alpha}_{\mathsf{E'}}}.$$

Proof. Similarly to the argument which leads to Equation (31), the fact that the column tessellation $\mathcal{Y}$ has only horizontal and vertical plates gives us

$$\gamma_{\mathsf{P}}\bar{a}_{\mathsf{P}} = \gamma_{\mathsf{P[hor]}}\bar{a}_{\mathsf{P[hor]}} + \gamma_{\mathsf{P[vert]}}\bar{a}_{\mathsf{P[vert]}}.$$

For each cell z'_j of the stationary random planar tessellation $\mathcal{Y}'$, we mark its circumcenter $c(\mathsf{z}'_j)$ with $\mathsf{z}'_{jo} = \mathsf{z}'_j - c(\mathsf{z}'_j)$ and Φ_j. We obtain the marked point process $\widetilde{\Phi}$ in the product space $\mathbb{R}^2 \times \mathscr{P}_2^o \times \mathscr{N}_s$. We notice that $\widetilde{\Phi}$ is already introduced in the proof of Proposition 2.2.1(i). Since the horizontal plates of $\mathcal{Y}$ in the column based on a cell z'_j of $\mathcal{Y}'$ are translations of z'_j, it is not difficult to see that

$$\gamma_{\mathsf{P[hor]}}\bar{a}_{\mathsf{P[hor]}} = \int \sum_{(c(z'_j), z'_{jo}, \varphi_j) \in \widetilde{\varphi}} \ \sum_{c \in \frac{1}{\rho_{z'_{jo}}}\varphi_j + c(z'_j)} \mathbf{1}_{[0,1]^3}(c) a(z'_{jo}) \mathbb{P}_{\widetilde{\Phi}}(\mathrm{d}\widetilde{\varphi})$$

$$= \int_{\mathbb{R}^2 \times \mathscr{P}_2^o \times \mathscr{N}_s} \sum_{c \in \frac{1}{\rho_{z'_o}}\varphi + c'} \mathbf{1}_{[0,1]^3}(c) a(z'_o) \widetilde{\Theta}(\mathrm{d}(c', z'_o, \varphi)),$$

where $\widetilde{\Theta}$ is the intensity measure of the marked point process $\widetilde{\Phi}$. This leads to

$$\gamma_{\mathsf{P[hor]}}\bar{a}_{\mathsf{P[hor]}} = \gamma_{\mathsf{Z'}} \int_{\mathscr{P}_2^o \times \mathscr{N}_s} \int_{\mathbb{R}^2} \sum_{c \in \frac{1}{\rho_{z'_o}}\varphi + c'} \mathbf{1}_{[0,1]^3}(c) a(z'_o) \lambda_2(\mathrm{d}c') \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)).$$

Recall that $\widetilde{\mathbb{Q}}$ is the joint distribution of the two marks z'_o and Φ of the typical cell-circumcenter of the planar tessellation $\mathcal{Y}'$. Write $c := (c_1, c_2, c_3)$. Then

$$\gamma_{\mathsf{P[hor]}}\bar{a}_{\mathsf{P[hor]}} = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o \times \mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}}\varphi} \mathbf{1}_{[0,1]}(c_3) a(z'_o) \widetilde{\mathbb{Q}}(\mathrm{d}(z'_o, \varphi)).$$

Recall that Φ has distribution $\mathbb{Q}_1$. Moreover, $\mathbb{Q}'_1$ is the grain distribution of Z'. Since Φ is independent of $\mathcal{Y}'$, we infer that

$$\gamma_{\mathsf{P[hor]}}\bar{a}_{\mathsf{P[hor]}} = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o} a(z'_o) \int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_o}}\varphi} \mathbf{1}_{[0,1]}(c_3) \mathbb{Q}_1(\mathrm{d}\varphi) \mathbb{Q}'_1(\mathrm{d}z'_o) =$$

$$\gamma_{\mathsf{Z}'} \int\limits_{\mathscr{P}_2^o} a(z'_o)\rho_{z'_o} \mathbb{Q}'_1(\mathrm{d}z'_o) = \gamma_{\mathsf{Z}'} \int\limits_{\mathscr{T}'} a(z'_o(T'))\rho_{z'_o(T')} \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') = \gamma_{\mathsf{Z}'} \mathbb{E}_{\mathsf{Z}'}(a(\mathsf{z}')\rho_{\mathsf{z}'}) = \gamma_{\mathsf{Z}'}\bar{\theta}_{\mathsf{Z}'}.$$

In order to determine $\gamma_{\mathsf{P[vert]}}\bar{a}_{\mathsf{P[vert]}}$, for each edge e'_j with two adjacent cells z'_{j1} and z'_{j2} of $\mathcal{Y}'$, we mark its circumcenter $c(\mathsf{e}'_j)$ with $\mathsf{e}'_{jo} = \mathsf{e}'_j - c(\mathsf{e}'_j)$, the two shifted adjacent cells $\mathsf{z}'_{j1} - c(\mathsf{e}'_j), \mathsf{z}'_{j2} - c(\mathsf{e}'_j)$ and the corresponding independent point processes Φ_{j1}, Φ_{j2}. We obtain the marked point process $\grave{\Phi}$ in the product space $\mathbb{R}^2 \times \mathscr{P}_1^o \times (\mathscr{P}_2)^2 \times \mathscr{N}_s^2$. Note that $\grave{\Phi}$ is already defined in the proof of Equation (32). Therefore

$$\gamma_{\mathsf{P[vert]}}\bar{a}_{\mathsf{P[vert]}} =$$

$$= \int \sum_{(c(e'_j), e'_{jo}, z'_{j1} - c(e'_j), z'_{j2} - c(e'_j), \varphi_{j1}, \varphi_{j2}) \in \grave{\varphi}} \; \sum_{c \in \frac{1}{\rho_{z'_{j1} - c(e'_j)}}\varphi_{j1} \cup \frac{1}{\rho_{z'_{j2} - c(e'_j)}}\varphi_{j2} + c(e'_j)} \mathbf{1}_{[0,1]^3}(c) \cdot a_c \mathbb{P}_{\grave{\Phi}}(\mathrm{d}\grave{\varphi})$$

$$= \int\limits_{\mathbb{R}^2 \times \mathscr{P}_1^o \times (\mathscr{P}_2)^2 \times \mathscr{N}_s^2} \sum_{c \in \frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)} + c'} \mathbf{1}_{[0,1]^3}(c) \cdot a_c \, \grave{\Theta}(\mathrm{d}(c', e'_o, z'_{(1)}, z'_{(2)}, \varphi_{(1)}, \varphi_{(2)})),$$

where for each $c \in \frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)} + c'$, the notation a_c is the area of the vertical rectangle whose lower horizontal side possesses c as midpoint and whose upper horizontal side possesses the upper consecutive point of c (which also belongs to $\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)} + c'$) as midpoint. Here, $\grave{\Theta}$ is the intensity measure of the marked point process $\grave{\Phi}$. So we get

$$\gamma_{\mathsf{P[vert]}}\bar{a}_{\mathsf{P[vert]}} = \gamma_{\mathsf{E}'} \int\limits_{\mathscr{P}_1^o \times (\mathscr{P}_2)^2} \int\limits_{\mathscr{N}_s} \int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3) a_{(0,0,c_3)}$$

$$\mathbb{Q}_1(\mathrm{d}\varphi_{(1)}) \mathbb{Q}_1(\mathrm{d}\varphi_{(2)}) \mathbb{Q}'_3(\mathrm{d}(e'_o, z'_{(1)}, z'_{(2)}))$$

$$= \gamma_{\mathsf{E}'} \int\limits_{\mathscr{P}_1^o \times (\mathscr{P}_2)^2} \ell(e'_o) \int\limits_{\mathscr{N}_s} \int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3) \ell_{(0,0,c_3)}$$

$$\mathbb{Q}_1(\mathrm{d}\varphi_{(1)})\mathbb{Q}_1(\mathrm{d}\varphi_{(2)})\mathbb{Q}_3'(\mathrm{d}(e_o', z_{(1)}', z_{(2)}'))$$

where $\ell_{(0,0,c_3)}$ is the distance from $(0,0,c_3) \in \frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}$ to the upper consecutive point of $(0,0,c_3)$ also belonging to $\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}$. Furthermore, $a_{(0,0,c_3)}$ is the area of the vertical rectangle whose lower horizontal side possesses $(0,0,c_3) \in \frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}$ as midpoint and whose upper horizontal side possesses the upper consecutive point of $(0,0,c_3)$ (also belonging to $\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)} \cup \frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}$) as midpoint. Similarly to Equation (35), we get

$$\int\limits_{\mathscr{N}_s}\int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3)\in\frac{1}{\rho_{z'_{(1)}}}\varphi_{(1)}\cup\frac{1}{\rho_{z'_{(2)}}}\varphi_{(2)}} \mathbf{1}_{[0,1]}(c_3)\ell_{(0,0,c_3)}\mathbb{Q}_1(\mathrm{d}\varphi_{(1)})\mathbb{Q}_1(\mathrm{d}\varphi_{(2)}) = 1,$$

which yields

$$\begin{aligned}\gamma_{\mathsf{P[vert]}}\bar{a}_{\mathsf{P[vert]}} &= \gamma_{\mathsf{E'}} \int\limits_{\mathscr{P}_1^o\times(\mathscr{P}_2)^2} \ell(e_o')\mathbb{Q}_3'(\mathrm{d}(e_o', z_{(1)}', z_{(2)}')) = \gamma_{\mathsf{E'}}\int\limits_{\mathscr{T}'} \ell(e_o'(T'))\mathbb{Q}_{\mathsf{E'}}(\mathrm{d}T') \\ &= \gamma_{\mathsf{E'}}\mathbb{E}_{\mathsf{E'}}(\ell(\mathsf{e'})) = \gamma_{\mathsf{E'}}\bar{\ell}_{\mathsf{E'}}.\end{aligned}$$

Therefore, using Proposition 2.2.1(iv) for the intensity γ_{P} of plates in $\mathcal{Y}$, we obtain

$$\bar{a}_{\mathsf{P}} = \frac{\gamma_{\mathsf{P[hor]}}\bar{a}_{\mathsf{P[hor]}} + \gamma_{\mathsf{P[vert]}}\bar{a}_{\mathsf{P[vert]}}}{\gamma_{\mathsf{P}}} = \frac{\gamma_{\mathsf{Z'}}\bar{\theta}_{\mathsf{Z'}} + \gamma_{\mathsf{E'}}\bar{\ell}_{\mathsf{E'}}}{\gamma_{\mathsf{Z'}}\bar{\rho}_{\mathsf{Z'}} + \gamma_{\mathsf{E'}}\bar{\alpha}_{\mathsf{E'}}} = \frac{(\mu_{\mathsf{V'E'}}-2)\bar{\theta}_{\mathsf{Z'}} + \mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}}}{(\mu_{\mathsf{V'E'}}-2)\bar{\rho}_{\mathsf{Z'}} + \mu_{\mathsf{V'E'}}\bar{\alpha}_{\mathsf{E'}}}.$$

□

Remark 2.2.21. From the fact that each plate of a 3-dimensional tessellation is adjacent to 2 cells and 2 cell-facets, using mean value identities in [**39**], we arrive at

$$\gamma_{\mathsf{Z}}\bar{a}_{\mathsf{Z}} = \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{p:\ p\subset z}} a(\mathsf{p})\Big) = \gamma_{\mathsf{P}}\mathbb{E}_{\mathsf{P}}\Big(\sum_{\mathsf{z:\ z\supset p}} a(\mathsf{p})\Big) = 2\gamma_{\mathsf{P}}\bar{a}_{\mathsf{P}}$$

and analogously, $\gamma_{\mathsf{Z_2}}\bar{a}_{\mathsf{Z_2}} = 2\gamma_{\mathsf{P}}\bar{a}_{\mathsf{P}}$. With the help of Proposition 2.2.1(i) and Proposition 2.2.2(iv) we infer that

$$\begin{aligned}\bar{a}_{\mathsf{Z}} &= \frac{2\gamma_{\mathsf{P}}\bar{a}_{\mathsf{P}}}{\gamma_{\mathsf{Z}}} = \frac{2(\gamma_{\mathsf{Z'}}\bar{\theta}_{\mathsf{Z'}} + \gamma_{\mathsf{E'}}\bar{\ell}_{\mathsf{E'}})}{\gamma_{\mathsf{Z'}}\bar{\rho}_{\mathsf{Z'}}} = \frac{2}{\bar{\rho}_{\mathsf{Z'}}}\left(\bar{\theta}_{\mathsf{Z'}} + \frac{\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}}}{\mu_{\mathsf{V'E'}}-2}\right), \\ \bar{a}_{\mathsf{Z_2}} &= \frac{2\gamma_{\mathsf{P}}\bar{a}_{\mathsf{P}}}{\gamma_{\mathsf{Z_2}}} = \frac{2\gamma_{\mathsf{Z'}}\bar{\theta}_{\mathsf{Z'}} + 2\gamma_{\mathsf{E'}}\bar{\ell}_{\mathsf{E'}}}{2\gamma_{\mathsf{Z'}}\bar{\rho}_{\mathsf{Z'}} + \gamma_{\mathsf{V'}}\bar{\beta}_{\mathsf{V'}}} = \frac{(\mu_{\mathsf{V'E'}}-2)\bar{\theta}_{\mathsf{Z'}} + \mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}}}{(\mu_{\mathsf{V'E'}}-2)\bar{\rho}_{\mathsf{Z'}} + \bar{\beta}_{\mathsf{V'}}}.\end{aligned}$$

We can also consider mean values of type $\bar{a}_{\mathsf{XY}}$ - the mean total area of all Y-type objects adjacent to the typical X-type object, where dim(Y-type object)= 2. Again some equations are obvious:

$$\bar{a}_{\mathsf{ZP}} = \bar{a}_{\mathsf{Z}}, \quad \bar{a}_{\mathsf{Z_2P}} = \bar{a}_{\mathsf{Z_2}}.$$

It is interesting in this context to evaluate the mean total area of all facets adjacent to the typical cell, namely $\bar{a}_{\mathsf{ZZ_2}}$. In a facet-to-facet random spatial tessellation, it is easy to see that $\bar{a}_{\mathsf{ZZ_2}} = 2\bar{a}_{\mathsf{Z}}$, because each cell-facet is a plate and the multiset $\mathsf{Z_2}$

of cell-facets is equal to the set P of plates up to the multiplicity 2. It is difficult to determine $\bar{a}_{\mathsf{Z}\mathsf{Z}_2}$ for an arbitrary non-facet-to-facet random spatial tessellation; we only know that $\bar{a}_{\mathsf{Z}\mathsf{Z}_2} \geq \bar{a}_{\mathsf{Z}}$. Indeed, for an arbitrary cell z let $a_{\mathsf{Z}_2}(\mathsf{z})$ be the total area of all cell-facets adjacent to the cell z. We observe that $a_{\mathsf{Z}_2}(\mathsf{z})$ is the sum of the surface area of the cell z and the areas of facets of neighbouring cells adjacent to z. For the column tessellation $\mathcal{Y}$, however, we can compute $\bar{a}_{\mathsf{Z}\mathsf{Z}_2}$, using the fact that each horizontal facet of a cell is also a facet of one other cell and each vertical facet is an element of Z_2 with multiplicity 1. So any cell z of $\mathcal{Y}$ is adjacent to its facets, obviously, and to the two horizontal facets of the neighbouring cells within the same column. These two horizontal facets are identical to the base and top facet of z. There are no further facets adjacent to z. Therefore, denoting by $\mathsf{p}[\mathsf{hor}]$ some horizontal plate of $\mathcal{Y}$ and observing that $\mathsf{Z}_2^{\neq}[\mathsf{hor}] = \mathsf{P}[\mathsf{hor}]$, we get

$$\gamma_{\mathsf{Z}}\bar{a}_{\mathsf{Z}\mathsf{Z}_2} = \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{z}_2\in\mathsf{Z}_2:\ \mathsf{z}_2\subset\mathsf{z}} a(\mathsf{z}_2)\Big) = \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{z}_2\in\mathsf{Z}_2^{\neq}:(\mathsf{z}_2,\mathsf{z})\in b} a(\mathsf{z}_2) + \sum_{\mathsf{p}[\mathsf{hor}]:\ \mathsf{p}[\mathsf{hor}]\subset\mathsf{z}} a(\mathsf{p}[\mathsf{hor}])\Big)$$

$$= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}(a(\mathsf{z})) + \gamma_{\mathsf{P}[\mathsf{hor}]}\mathbb{E}_{\mathsf{P}[\mathsf{hor}]}\Big(a(\mathsf{p}[\mathsf{hor}])\sum_{\mathsf{z}:\ \mathsf{z}\supset\mathsf{p}[\mathsf{hor}]} 1\Big) = \gamma_{\mathsf{Z}}\bar{a}_{\mathsf{Z}} + 2\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{a}_{\mathsf{P}[\mathsf{hor}]}$$

$$= 2(\gamma_{\mathsf{Z}'}\bar{\theta}_{\mathsf{Z}'} + \gamma_{\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}) + 2\gamma_{\mathsf{Z}'}\bar{\theta}_{\mathsf{Z}'} = 2\gamma_{\mathsf{Z}'}\left(2\bar{\theta}_{\mathsf{Z}'} + \frac{\gamma_{\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\gamma_{\mathsf{Z}'}}\right) = 2\gamma_{\mathsf{Z}'}\left(2\bar{\theta}_{\mathsf{Z}'} + \frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'} - 2}\right),$$

hence

$$\bar{a}_{\mathsf{Z}\mathsf{Z}_2} = \frac{\gamma_{\mathsf{Z}}\bar{a}_{\mathsf{Z}\mathsf{Z}_2}}{\gamma_{\mathsf{Z}}} = \frac{2\gamma_{\mathsf{Z}'}}{\gamma_{\mathsf{Z}'}\bar{\rho}_{\mathsf{Z}'}}\left(2\bar{\theta}_{\mathsf{Z}'} + \frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'} - 2}\right) = \frac{2}{\bar{\rho}_{\mathsf{Z}'}}\left(2\bar{\theta}_{\mathsf{Z}'} + \frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'} - 2}\right).$$

We present results involving volumes in the next theorem.

Theorem 2.2.22. *The mean volume of the typical cell of the column tessellation, denoted by $\bar{v}_{\mathsf{Z}}$, is*

$$\bar{v}_{\mathsf{Z}} = \frac{1}{\gamma_{\mathsf{Z}'}\bar{\rho}_{\mathsf{Z}'}}.$$

Proof. It is obvious from the fact that $\gamma_{\mathsf{Z}}\bar{v}_{\mathsf{Z}} = 1$; see [**3**, Section 9.4]. □

Corollary 2.2.23. *The corresponding area and volume mean values of column tessellations of constant cell-height* 1 *are*

$$\bar{a}_{\mathsf{P}} = \frac{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\bar{a}_{\mathsf{Z}'} + \mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{3\mu_{\mathsf{V}'\mathsf{E}'} - 2}, \quad \bar{a}_{\mathsf{Z}} = 2\left(\bar{a}_{\mathsf{Z}'} + \frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'} - 2}\right),$$

$$\bar{a}_{\mathsf{Z}_2} = \frac{(\mu_{\mathsf{V}'\mathsf{E}'} - 2)\bar{a}_{\mathsf{Z}'} + \mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{2\mu_{\mathsf{V}'\mathsf{E}'} - \phi - 2}, \quad \bar{a}_{\mathsf{Z}\mathsf{Z}_2} = 2\left(2\bar{a}_{\mathsf{Z}'} + \frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'} - 2}\right),$$

$$\bar{v}_{\mathsf{Z}} = \frac{1}{\gamma_{\mathsf{Z}'}}.$$

Here, $\bar{a}_{\mathsf{Z}'}$ is the mean area of the typical planar cell.

Proof. We emphasize that the mean area $\bar{a}_{\mathsf{Z}'}$ of the typical planar cell is not a new parameter because $\bar{a}_{\mathsf{Z}'} = 1/\gamma_{\mathsf{Z}'} = 2/[\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)]$. The results come directly from the fact that in the case of a column tessellation with height 1 we have $\overline{\rho}_{\mathsf{Z}'} = 1$, $\overline{\alpha}_{\mathsf{E}'} = 2$, $\overline{\theta}_{\mathsf{Z}'} = \bar{a}_{\mathsf{Z}'}$ and $\overline{\beta}_{\mathsf{V}'} = \mu_{\mathsf{V}'\mathsf{E}'} - \phi$; see Remark 2.1.8. $\square$

Remark 2.2.24. For $\mathsf{z}' \in \mathcal{Y}'$ the function $\rho_{\mathsf{z}'}$ may depend not only on the polygonal cell z' but also on the whole tessellation $\mathcal{Y}'$. In particular, $\rho_{\mathsf{z}'}$ is a function of some aspects of $\mathcal{Y}'$ viewed from z' such as

$$\rho_{\mathsf{z}'} = m_{\mathsf{V}'}(\mathsf{z}'), \quad \rho_{\mathsf{z}'} = \sum_{\widetilde{\mathsf{z}}' :\, \widetilde{\mathsf{z}}' \cap \mathsf{z}' \neq \emptyset} a(\widetilde{\mathsf{z}}'), \quad \rho_{\mathsf{z}'} = \sum_{\mathsf{e}' :\, \dim(\mathsf{e}' \cap \mathsf{z}') = 0} \ell(\mathsf{e}'),$$

noting that $\dim(\mathsf{e}' \cap \mathsf{z}') = 0$ implies that $\mathsf{e}' \cap \mathsf{z}' \neq \emptyset$. We present for example the computation of the cell-intensity γ_{Z} of $\mathcal{Y}$ for the case $\rho_{\mathsf{z}'} = m_{\mathsf{V}'}(\mathsf{z}')$. Recalling that we choose the reference point of a spatial cell $\mathsf{z} \in \mathcal{Y}$ as the circumcenter of the base facet of z, we obtain

$$\gamma_{\mathsf{Z}} = \mathbb{E} \sum_{\mathsf{z}'_j \in \mathsf{Z}'} \sum_{\mathsf{c} \in \frac{1}{m_{\mathsf{V}' - c(\mathsf{z}'_j)}(\mathsf{z}'_{jo})} \Phi_j + c(\mathsf{z}'_j)} \mathbf{1}_{[0,1]^3}(\mathsf{c}),$$

where $\mathsf{z}'_{jo} = \mathsf{z}'_j - c(\mathsf{z}'_j)$ for $\mathsf{z}'_j \in \mathsf{Z}'$. Therefore, for $j = 1, 2, \ldots$, using the independence of Φ_j – the stationary simple point process on $\{0\}^2 \times \mathbb{R}$ with intensity 1 – and $\mathcal{Y}'$, we obtain

$$\gamma_{\mathsf{Z}} = \int_{\mathscr{T}'} \sum_{z'_j \in Z'(y')} \int_{\mathscr{N}_s} \sum_{c \in \frac{1}{m_{V'(y') - c(z'_j)}(z'_{jo})} \varphi_j + c(z'_j)} \mathbf{1}_{[0,1]^3}(c) \mathbb{Q}_1(\mathrm{d}\varphi_j) \mathbb{P}_{\mathcal{Y}'}(\mathrm{d}y'),$$

where $Z'(y')$ and $V'(y')$ are the sets of cells and vertices of a realization y' of the stationary random planar tessellation $\mathcal{Y}'$, respectively. Recall that $\mathbb{Q}_1$ is the distribution of Φ_1. Here, $\mathbb{P}_{\mathcal{Y}'}$ denotes the distribution of $\mathcal{Y}'$ and $m_{V'(y') - c(z'_j)}(z'_{jo})$ is the number of vertices of $y' - c(z'_j)$ on the boundary of the cell z'_{jo}. Recall that for a planar tessellation T', $z'_o(T')$ is the cell of T' with circumcenter at the origin o if such a cell exists. Otherwise, $z'_o(T') = \emptyset$. Let $V'(T')$ be the set of vertices of T'. Now, Equation (22) gives us

$$\gamma_{\mathsf{Z}} = \gamma_{\mathsf{Z}'} \int_{\mathscr{T}'} \int_{\mathbb{R}^2} \int_{\mathscr{N}_s} \sum_{c \in \frac{1}{m_{V'(T')}(z'_o(T'))} \varphi + c'} \mathbf{1}_{[0,1]^3}(c) \mathbb{Q}_1(\mathrm{d}\varphi) \lambda_2(\mathrm{d}c') \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T').$$

Similarly to the proof of Proposition 2.2.1(i), we arrive at

$$\begin{aligned}
\gamma_{\mathsf{Z}} &= \gamma_{\mathsf{Z}'} \int_{\mathscr{T}'} \int_{\mathscr{N}_s} \sum_{(0,0,c_3) \in \frac{1}{m_{V'(T')}(z'_o(T'))} \varphi} \mathbf{1}_{[0,1]}(c_3) \mathbb{Q}_1(\mathrm{d}\varphi) \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') \\
&= \gamma_{\mathsf{Z}'} \int_{\mathscr{T}'} m_{V'(T')}(z'_o(T')) \mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') \\
&= \gamma_{\mathsf{Z}'} \mathbb{E}_{\mathsf{Z}'}(m_{\mathsf{V}'}(\mathsf{z}')) = \gamma_{\mathsf{Z}'} \mu_{\mathsf{Z}'\mathsf{V}'} = \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{Z}'} = \gamma_{\mathsf{V}'} \mu_{\mathsf{V}'\mathsf{E}'}.
\end{aligned}$$

Conjecture 2.2.25. *All the intensities, topological/interior parameters and metric mean values of the column tessellation* $\mathcal{Y}$ *do not change if for* $\mathsf{z}' \in \mathcal{Y}'$ *we take the function* $\rho_{\mathsf{z}'}$ *depending on the whole stationary random planar tessellation* $\mathcal{Y}'$*, in particular,* $\rho_{\mathsf{z}'}$ *is a function of some aspects of* $\mathcal{Y}'$ *viewed from* z'*.*

2.3. Three examples

In this section we will give three examples for column tessellations. The generating random planar tessellations are the Poisson line tessellation (PLT), the STIT tessellation and the Poisson-Voronoi tessellation (PVT), respectively. We will consider the column tessellations with constant cell height 1 ($\rho_{\mathsf{Z}'} = 1$) and we restrict for the random planar tessellations to the stationary and isotropic case.

The intensities, topological/interior parameters and metric mean values for those column tessellations are presented in Table 2. To facilitate the comparability of the results we assume that all the underlying random planar tessellations have the same cell-intensity $\gamma_{\mathsf{Z}'}$. In Table 1 the seven necessary parameters of the planar PLT, STIT and PVT are given, see [**23, 3**]. In a PLT all vertices have 4 emanating edges ($\mu_{\mathsf{V}'\mathsf{E}'} = 4$, $\mu^{(2)}_{\mathsf{V}'\mathsf{E}'} = 16$), whereas in STIT and PVT all vertices have 3 emanating edges ($\mu_{\mathsf{V}'\mathsf{E}'} = 3$, $\mu^{(2)}_{\mathsf{V}'\mathsf{E}'} = 9$). PLT and PVT are side-to-side ($\phi = \mu_{\mathsf{E}'\mathsf{V}'[\pi]} = 0$); a STIT tessellation is not side-to-side and all vertices are π-vertices ($\phi = 1$, $\mu_{\mathsf{E}'\mathsf{V}'[\pi]} = 2$). Throughout the chapter all our results were considered for the special case of a column tessellation with constant height 1. In Remark 2.1.8 the α-, β- and θ-mean values are given, Corollary 2.2.10 presents the intensities and topological/interior mean values and the metric mean values are given in Subsection 2.2.3. Using those results, the entries in Table 2 and other interesting quantities of those column tessellations can be computed.

$\mathcal{Y}'$	PLT	STIT	PVT
$\gamma_{\mathsf{V}'}$	$\gamma_{\mathsf{Z}'}$	$2\gamma_{\mathsf{Z}'}$	$2\gamma_{\mathsf{Z}'}$
$\mu_{\mathsf{V}'\mathsf{E}'}$	4	3	3
$\mu^{(2)}_{\mathsf{V}'\mathsf{E}'}$	16	9	9
ϕ	0	1	0
$\mu_{\mathsf{E}'\mathsf{V}'[\pi]}$	0	2	0
$\bar{\ell}_{\mathsf{E}'}$	$\frac{\sqrt{\pi}}{2\sqrt{\gamma_{\mathsf{Z}'}}}$	$\frac{\sqrt{\pi}}{3\sqrt{\gamma_{\mathsf{Z}'}}}$	$\frac{2}{3\sqrt{\gamma_{\mathsf{Z}'}}}$
$\bar{a}_{\mathsf{Z}'}$	$\frac{1}{\gamma_{\mathsf{Z}'}}$	$\frac{1}{\gamma_{\mathsf{Z}'}}$	$\frac{1}{\gamma_{\mathsf{Z}'}}$

TABLE 1. Seven parameters of the random planar tessellation.

$\mathcal{Y}/\mathcal{Y}'$	PLT	STIT	PVT
γ_{V}	$4\gamma_{\mathsf{Z}'}$	$6\gamma_{\mathsf{Z}'}$	$6\gamma_{\mathsf{Z}'}$
γ_{P}	$5\gamma_{\mathsf{Z}'}$	$7\gamma_{\mathsf{Z}'}$	$7\gamma_{\mathsf{Z}'}$
μ_{PV}	$\frac{28}{5}$	$\frac{36}{7}$	$\frac{36}{7}$
μ_{EP}	$\frac{7}{2}$	3	3
ξ	$\frac{1}{2}$	1	$\frac{1}{2}$
κ	0	$\frac{2}{3}$	0
ψ	3	2	2
τ	2	$\frac{4}{3}$	1
$\bar{\ell}_{\mathsf{E}}$	$\frac{\sqrt{\pi}}{4\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{1}{8}$	$\frac{\sqrt{\pi}}{6\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{1}{6}$	$\frac{1}{3\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{1}{6}$
$\bar{\ell}_{\mathsf{P}}$	$\frac{6\sqrt{\pi}}{5\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{4}{5}$	$\frac{6\sqrt{\pi}}{7\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{6}{7}$	$\frac{12}{7\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{6}{7}$
$\bar{\ell}_{\mathsf{Z}}$	$\frac{4\sqrt{\pi}}{\sqrt{\gamma_{\mathsf{Z}'}}}+4$	$\frac{4\sqrt{\pi}}{\sqrt{\gamma_{\mathsf{Z}'}}}+4$	$\frac{8}{\sqrt{\gamma_{\mathsf{Z}'}}}+6$
$\bar{\ell}_{\mathsf{ZE}}$	$\frac{6\sqrt{\pi}}{\sqrt{\gamma_{\mathsf{Z}'}}}+4$	$\frac{6\sqrt{\pi}}{\sqrt{\gamma_{\mathsf{Z}'}}}+6$	$\frac{12}{\sqrt{\gamma_{\mathsf{Z}'}}}+6$
$\bar{\ell}_{\mathsf{Z_2E}}$	$\frac{5\sqrt{\pi}}{3\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{4}{3}$	$\frac{5\sqrt{\pi}}{3\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{5}{3}$	$\frac{5}{2\sqrt{\gamma_{\mathsf{Z}'}}}+\frac{3}{2}$
$\bar{a}_{\mathsf{P}}$	$\frac{1}{5\gamma_{\mathsf{Z}'}}+\frac{\sqrt{\pi}}{5\sqrt{\gamma_{\mathsf{Z}'}}}$	$\frac{1}{7\gamma_{\mathsf{Z}'}}+\frac{\sqrt{\pi}}{7\sqrt{\gamma_{\mathsf{Z}'}}}$	$\frac{1}{7\gamma_{\mathsf{Z}'}}+\frac{2}{7\sqrt{\gamma_{\mathsf{Z}'}}}$
$\bar{a}_{\mathsf{ZZ_2}}$	$\frac{4}{\gamma_{\mathsf{Z}'}}+\frac{2\sqrt{\pi}}{\sqrt{\gamma_{\mathsf{Z}'}}}$	$\frac{4}{\gamma_{\mathsf{Z}'}}+\frac{2\sqrt{\pi}}{\sqrt{\gamma_{\mathsf{Z}'}}}$	$\frac{4}{\gamma_{\mathsf{Z}'}}+\frac{4}{\sqrt{\gamma_{\mathsf{Z}'}}}$

TABLE 2. Fifteen mean values of the corresponding column tessellation with height 1.

2.4. Stratum tessellations

To conclude this chapter we determine the intensities, topological/interior parameters and metric mean values of a stratum tessellation. The last model was introduced in [**14**].

2.4.1. Construction. Based on the stationary random planar tessellation $\mathcal{Y}'$ in the horizontal plane $\mathcal{E} = \mathbb{R}^2 \times \{0\}$ we construct the spatial stratum tessellation $\widetilde{\mathcal{Y}}$ in the following way:

For each cell z' of $\mathcal{Y}'$, we consider an infinite cylindrical column based on this cell and perpendicular to $\mathcal{E}$. Now, we construct on $\{0\}^2 \times \mathbb{R}$ a stationary and simple point process Φ_γ with intensity $\gamma \in (0, \infty)$ which is independent of $\mathcal{Y}'$. To create the spatial tessellation, all the columns are intersected by horizontal planes going through each of the random points of Φ_γ. The resulting random three-dimensional tessellation $\widetilde{\mathcal{Y}}$ is called a *stratum tessellation*. Note that the stratum tessellation is facet-to-facet if the generating random planar tessellation is side-to-side. A stratum tessellation with constant height 1, which implies that $\gamma = 1$, is considered in Example 1.2.5.

2.4.2. Basic properties. For a vertex $\mathsf{v}' \in \mathsf{V}'$ in $\mathcal{Y}'$ we consider the vertex-line $\mathcal{L}_{\mathsf{v}'}$ through v'. The vertices of the stratum tessellation $\widetilde{\mathcal{Y}}$ on $\mathcal{L}_{\mathsf{v}'}$ form a stationary simple point process on $\mathcal{L}_{\mathsf{v}'}$, denoted by $\Phi_{\mathsf{v}'}$. It is easy to see that the intensity of $\Phi_{\mathsf{v}'}$ is the same as that of Φ_γ, which is equal to γ.

If v' is a non-π-vertex $\mathsf{v}'[\bar{\pi}]$, then each point of $\Phi_{\mathsf{v}'[\bar{\pi}]}$ is the apex of $2m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])$ cells. Beside, each point of $\Phi_{\mathsf{v}'[\bar{\pi}]}$ is also the 0-face of $m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])$ horizontal plates and $2m_{\mathsf{E}'}(\mathsf{v}'[\bar{\pi}])$ vertical plates.

If v' is a π-vertex $\mathsf{v}'[\pi]$ then each point of $\Phi_{\mathsf{v}'[\pi]}$ is the apex of $2(m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 1)$ cells and in the relative interior of 2 ridges of 2 other cells. Moreover, each point of $\Phi_{\mathsf{v}'[\pi]}$ is the 0-face of $m_{\mathsf{E}'}(\mathsf{v}'[\pi]) - 1$ horizontal plates and $2m_{\mathsf{E}'}(\mathsf{v}'[\pi])$ vertical plate and in the relative interior of 1 side of 1 horizontal plate.

Property 2.4.1. Let v' be a vertex in $\mathcal{Y}'$. Then the point process $\Phi_{\mathsf{v}'}$ has intensity γ and each point of $\Phi_{\mathsf{v}'}$ is adjacent to $2m_{\mathsf{E}'}(\mathsf{v}')$ cells and $3m_{\mathsf{E}'}(\mathsf{v}')$ plates of $\widetilde{\mathcal{Y}}$. Moreover, if v' is a π-vertex $\mathsf{v}'[\pi]$ then each vertex of $\widetilde{\mathcal{Y}}$ on $\mathcal{L}_{\mathsf{v}'[\pi]}$ is adjacent to 2 relative ridge-interiors and 1 relative plate-side-interior. If the vertex v' of $\mathcal{Y}'$ is a non-π-vertex $\mathsf{v}'[\bar{\pi}]$ then each vertex of $\widetilde{\mathcal{Y}}$ on $\mathcal{L}_{\mathsf{v}'[\bar{\pi}]}$ is not adjacent to any relative ridge-interiors or relative plate-side-interiors.

Furthermore, each edge e' of $\mathcal{Y}'$ is adjacent to two planar cells. We have a particle process of horizontal edges of $\widetilde{\mathcal{Y}}$ in the common face of the two neighbouring columns based on these two planar cells. This particle process, denoted by $\Phi_{\mathsf{e}'}$, has intensity γ. All the edges of $\Phi_{\mathsf{e}'}$ are translations of e'. Besides, for any $\mathsf{v}' \in \mathsf{V}'$, the intensity of the particle process of vertical edges of $\widetilde{\mathcal{Y}}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'}$ is γ.

Property 2.4.2. Let e' be an edge in $\mathcal{Y}'$. Then the particle process $\Phi_{\mathsf{e}'}$ has intensity γ. Any horizontal edge of $\widetilde{\mathcal{Y}}$ has four emanating plates, two of them are horizontal and the other two are vertical. Any horizontal edge of $\widetilde{\mathcal{Y}}$ is adjacent to four cells.

Let v' be a vertex in $\mathcal{Y}'$. The intensity of the particle process of vertical edges of $\widetilde{\mathcal{Y}}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'}$ is γ. Each particle (vertical edge) of this process accepts v' as its corresponding vertex in $\mathcal{Y}'$ and is adjacent to $m_{\mathsf{E}'}(\mathsf{v}')$ cells and $m_{\mathsf{E}'}(\mathsf{v}')$ plates of $\widetilde{\mathcal{Y}}$.

In the next subsections, in order to compute the intensities, topological/interior parameters and metric mean values of the stratum tessellation $\widetilde{\mathcal{Y}}$ we use a similar method as for the column tessellation $\mathcal{Y}$.

2.4.3. Formulae for intensities.

Proposition 2.4.3. *The intensities of primitive elements of the stratum tessellation $\widetilde{\mathcal{Y}}$ depends on $\mathcal{Y}'$ and γ as follows:*

$$\gamma_{\mathsf{V}} = \gamma\gamma_{\mathsf{V}'},\ \gamma_{\mathsf{E}} = \frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} + \gamma\gamma_{\mathsf{V}'},\ \gamma_{\mathsf{P}} = \gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma\gamma_{\mathsf{V}'},\ \gamma_{\mathsf{Z}} = \frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma\gamma_{\mathsf{V}'}.$$

For a refined partition of the sets E and P of $\widetilde{\mathcal{Y}}$ into horizontal and vertical elements we obtain

$$\gamma_{\mathsf{E[hor]}} = \frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'},\ \gamma_{\mathsf{E[vert]}} = \gamma\gamma_{\mathsf{V}'},\ \gamma_{\mathsf{P[hor]}} = \frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - \gamma\gamma_{\mathsf{V}'},\ \gamma_{\mathsf{P[vert]}} = \frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}.$$

For those intensities we need the intensity $\gamma_{\mathsf{V}'}$ and the mean value $\mu_{\mathsf{V}'\mathsf{E}'}$ of $\mathcal{Y}'$.

Proof. For instance, the vertex-intensity γ_{V} of the stratum tessellation $\widetilde{\mathcal{Y}}$ is calculated in the following way: We mark each vertex v'_j of the random planar tessellation $\mathcal{Y}'$ with the stationary simple point process Φ_γ. We obtain a marked point process, denoted by $\widehat{\Psi}$, in the product space $\mathbb{R}^2 \times \mathscr{N}_s$. Let $\mathbb{P}_{\widehat{\Psi}}$ and $\mathbb{Q}_\gamma$ be the distribution of $\widehat{\Psi}$ and that of Φ_γ, respectively. We have, using Theorem 1.1.15,

$$\begin{aligned}
\gamma_{\mathsf{V}} &= \int\limits_{\mathbb{R}^2\times\mathscr{N}_s} \sum_{(v'_j,\varphi_\gamma)\in\widehat{\psi}} \sum_{v\in\varphi_\gamma+v'_j} \mathbf{1}_{[0,1]^3}(v)\mathbb{P}_{\widehat{\Psi}}(\mathrm{d}\widehat{\psi}) \\
&= \gamma_{\mathsf{V}'} \int\limits_{\mathscr{N}_s}\int\limits_{\mathbb{R}^2} \sum_{v\in\varphi_\gamma+v'} \mathbf{1}_{[0,1]^3}(v)\lambda_2(\mathrm{d}v')\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma) \\
&= \gamma_{\mathsf{V}'} \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3)\in\varphi_\gamma} \mathbf{1}_{[0,1]}(v_3)\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma) = \gamma\gamma_{\mathsf{V}'}.
\end{aligned}$$

Similar arguments work for the other intensities. □

Proposition 2.4.4. *The intensity γ_{P_1} of plate-sides, the intensity γ_{Z_0} of cell-apices, the intensity γ_{Z_1} of cell-ridges and the intensity γ_{Z_2} of cell-facets of the stratum tessellation are given as follows*

$$\begin{gathered}
\gamma_{\mathsf{P}_1} = \gamma\gamma_{\mathsf{V}'}(3\mu_{\mathsf{V}'\mathsf{E}'} - \phi), \quad \gamma_{\mathsf{Z}_0} = 2\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi), \\
\gamma_{\mathsf{Z}_1} = 3\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - \phi), \quad \gamma_{\mathsf{Z}_2} = \gamma\gamma_{\mathsf{V}'}(2\mu_{\mathsf{V}'\mathsf{E}'} - \phi - 2).
\end{gathered}$$

Note that for the calculation of those intensities, the interior parameter ϕ of the random planar tessellation is a necessary additional input.

Proof. In order to determine the plate-side-intensity γ_{P_1}, we observe that

$$\gamma_{\mathsf{P}_1} = \gamma_{(\mathsf{P[hor]})_1} + \gamma_{(\mathsf{P[vert]})_1} = \gamma_{(\mathsf{P[hor]})_0} + \gamma_{(\mathsf{P[vert]})_1}.$$

Similar to the proof of Proposition 2.2.2(i), we have $\gamma_{(\mathsf{P[vert]})_1} = 4\gamma_{\mathsf{P[vert]}}$. On the other hand, we mark each vertex $\mathsf{v}'_j \in \mathsf{V}'$ with Φ_γ and $\mathsf{N}_j = n_{\mathsf{Z}'}(\mathsf{v}'_j)$ – the random number of its owner cells. We obtain a marked point process, denoted by $\overline{\Psi}$, in the product space $\mathbb{R}^2 \times \mathscr{N}_s \times \mathbb{N}$. Let $\mathbb{P}_{\overline{\Psi}}$ be the distribution of $\overline{\Psi}$. We observe that $\mathsf{P[hor]})_0$ is a multiset and $(\mathsf{P[hor]})_0^{\neq} = \mathsf{V}$. For any vertex $\mathsf{v}'_j \in \mathsf{V}'$, we notice that each vertex v of the stratum tessellation $\widetilde{\mathcal{Y}}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'_j}$ satisfies that $n_{\mathsf{P[hor]}}(\mathsf{v}) = n_{\mathsf{Z}'}(\mathsf{v}'_j)$.

Consequently, using mean value identities in [**39**], Theorem 1.1.15, the law of total probability and the independence of Φ_γ and $\mathcal{Y}'$, we find that

$$\begin{aligned}\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = \gamma_{\mathsf{V}}\mathbb{E}_{\mathsf{V}}(n_{\mathsf{P}[\mathsf{hor}]}(\mathsf{v})) &= \int\limits_{\mathbb{R}^2\times\mathcal{N}_s\times\mathbb{N}} \sum_{(v_j',\varphi_\gamma,n_j)\in\overline{\psi}} \sum_{v\in\varphi_\gamma+v_j'} \mathbf{1}_{[0,1]^3}(v)n_j\,\mathbb{P}_{\overline{\Psi}}(\mathrm{d}\overline{\psi})\\
&= \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\int\limits_{\mathcal{N}_s}\int\limits_{\mathbb{R}^2}\sum_{v\in\varphi_\gamma+v'}\mathbf{1}_{[0,1]^3}(v)n\,\lambda_2(\mathrm{d}v')\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)\\
&= \gamma\gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}n\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\\
&= \gamma\gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')) = \gamma\gamma_{\mathsf{V}'}\nu_{\mathsf{V}'\mathsf{Z}'} = \gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi),\end{aligned}$$

recalling Equation (22) for the definition of $\mathbb{Q}_{\mathsf{V}'}$ and Remark 2.1.8 for $\nu_{\mathsf{V}'\mathsf{Z}'}$. Consequently $\gamma_{\mathsf{P}_1} = \gamma_{(\mathsf{P}[\mathsf{hor}])_0} + 4\gamma_{\mathsf{P}[\mathsf{vert}]} = \gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi) + 2\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} = \gamma\gamma_{\mathsf{V}'}(3\mu_{\mathsf{V}'\mathsf{E}'}-\phi)$. It is easy to see that

$$\gamma_{\mathsf{Z}_0} = 2\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = 2\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi).$$

If the reference point of a vertical cell-ridge of $\widetilde{\mathcal{Y}}$ is its lower endpoint which is a 0-face of a horizontal plate then

$$\gamma_{\mathsf{Z}_1} = \gamma_{\mathsf{Z}_1[\mathsf{hor}]} + \gamma_{\mathsf{Z}_1[\mathsf{vert}]} = 2\gamma_{(\mathsf{P}[\mathsf{hor}])_1} + \gamma_{(\mathsf{P}[\mathsf{hor}])_0} = 3\gamma_{(\mathsf{P}[\mathsf{hor}])_0} = 3\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi).$$

Each vertical cell-facet of $\widetilde{\mathcal{Y}}$ is a vertical rectangle with 4 sides, two of them are horizontal and two of them are vertical. We choose the reference point of a vertical cell-facet of $\widetilde{\mathcal{Y}}$ as the midpoint of its lower horizontal side which is also a side of a horizontal plate. Therefore

$$\begin{aligned}\gamma_{\mathsf{Z}_2} = \gamma_{\mathsf{Z}_2[\mathsf{hor}]} + \gamma_{\mathsf{Z}_2[\mathsf{vert}]} = 2\gamma_{\mathsf{P}[\mathsf{hor}]} + \gamma_{(\mathsf{P}[\mathsf{hor}])_1} &= \gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} - 2\gamma\gamma_{\mathsf{V}'} + \gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi)\\
&= \gamma\gamma_{\mathsf{V}'}(2\mu_{\mathsf{V}'\mathsf{E}'}-\phi-2).\end{aligned}$$

□

2.4.4. Formulae for the topological and interior parameters.

Theorem 2.4.5. *The three topological and four interior parameters of the stratum tessellation $\widetilde{\mathcal{Y}}$ are given as follows*

$$\mu_{\mathsf{VE}} = \mu_{\mathsf{V}'\mathsf{E}'}+2,\quad \mu_{\mathsf{EP}} = \frac{6\mu_{\mathsf{V}'\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'}+2},\quad \mu_{\mathsf{PV}} = \frac{3\mu_{\mathsf{V}'\mathsf{E}'}}{\mu_{\mathsf{V}'\mathsf{E}'}-1},$$

$$\xi = \frac{2\phi}{\mu_{\mathsf{V}'\mathsf{E}'}+2},\quad \kappa = 0,\quad \psi = 2\phi,\quad \tau = \phi.$$

Proof. For any vertex $\mathsf{v}' \in \mathsf{V}'$, we notice that each vertex v of the stratum tessellation $\widetilde{\mathcal{Y}}$ on the vertex-line $\mathcal{L}_{\mathsf{v}'}$ satisfies that $m_{\mathsf{E}[\mathsf{hor}]}(\mathsf{v}) = m_{\mathsf{E}'}(\mathsf{v}')$. Hence $\mu_{\mathsf{VE}} = \mu_{\mathsf{VE}[\mathsf{hor}]} + \mu_{\mathsf{VE}[\mathsf{vert}]} = \mu_{\mathsf{V}'\mathsf{E}'}+2$.

For μ_{EP}: Use Property 2.4.2.

For μ_{PV}: Use Property 2.4.1.

For ξ: Note firstly that all horizontal edges of $\widetilde{\mathcal{Y}}$ are not π-edges and secondly that a vertical edge of $\widetilde{\mathcal{Y}}$ is a π-edge if its corresponding vertex $\mathsf{v}' \in \mathcal{Y}'$ is a π-vertex.

For κ: Obviously because $\widetilde{\mathcal{Y}}$ does not have any hemi-vertices.

For ψ: Use Property 2.4.1.

For τ: Use Property 2.4.1. □

Proposition 2.4.6. *The mean numbers of vertices, edges, apices (0-faces) and ridges (1-faces), respectively, of the typical cell in $\widetilde{\mathcal{Y}}$ are*

$$\mu_{\mathsf{ZV}} = \frac{4\mu_{\mathsf{V'E'}}}{\mu_{\mathsf{V'E'}} - 2}, \quad \mu_{\mathsf{ZE}} = \frac{6\mu_{\mathsf{V'E'}}}{\mu_{\mathsf{V'E'}} - 2}, \quad \nu_0(\mathsf{Z}) = \frac{4(\mu_{\mathsf{V'E'}} - \phi)}{\mu_{\mathsf{V'E'}} - 2}, \quad \nu_1(\mathsf{Z}) = \frac{6(\mu_{\mathsf{V'E'}} - \phi)}{\mu_{\mathsf{V'E'}} - 2}.$$

Proof. Similar to Proposition 2.2.9. □

2.4.5. Formulae for the metric mean values.

Theorem 2.4.7. *For the mean lengths of 1-dimensional objects in the stratum tessellation $\widetilde{\mathcal{Y}}$, we have, using the mean length $\bar{\ell}_{\mathsf{E'}}$ of the typical edge of $\mathcal{Y}'$,*

$$\bar{\ell}_{\mathsf{E}} = \frac{\gamma\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + 2}{\gamma(\mu_{\mathsf{V'E'}} + 2)}, \quad \bar{\ell}_{\mathsf{P}} = \frac{\mu_{\mathsf{V'E'}}(2\gamma\bar{\ell}_{\mathsf{E'}} + 1)}{\gamma(\mu_{\mathsf{V'E'}} - 1)}, \quad \bar{\ell}_{\mathsf{Z}} = \frac{2(2\gamma\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + \mu_{\mathsf{V'E'}} - \phi)}{\gamma(\mu_{\mathsf{V'E'}} - 2)},$$

$$\bar{\ell}_{\mathsf{E}[\pi]} = \frac{1}{\gamma}, \quad \bar{\ell}_{\mathsf{Z}_1} = \frac{2\gamma\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + \mu_{\mathsf{V'E'}} - \phi}{3\gamma(\mu_{\mathsf{V'E'}} - \phi)}, \quad \bar{\ell}_{\mathsf{Z}_2} = \frac{2(2\gamma\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + \mu_{\mathsf{V'E'}} - \phi)}{\gamma(2\mu_{\mathsf{V'E'}} - \phi - 2)},$$

$$\bar{\ell}_{\mathsf{P}_1} = \frac{\mu_{\mathsf{V'E'}}(2\gamma\bar{\ell}_{\mathsf{E'}} + 1)}{\gamma(3\mu_{\mathsf{V'E'}} - \phi)}, \quad \bar{\ell}_{\mathsf{ZE}} = \frac{2\mu_{\mathsf{V'E'}}(2\gamma\bar{\ell}_{\mathsf{E'}} + 1)}{\gamma(\mu_{\mathsf{V'E'}} - 2)}, \quad \bar{\ell}_{\mathsf{Z}_2\mathsf{E}} = \frac{4\gamma\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + 2\mu_{\mathsf{V'E'}} - \phi}{\gamma(2\mu_{\mathsf{V'E'}} - \phi - 2)},$$

$$\bar{\ell}_{\mathsf{VE}} = \frac{\gamma\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}} + 2}{\gamma}, \quad \bar{\ell}_{\mathsf{ZP}} = \frac{4\mu_{\mathsf{V'E'}}(2\gamma\bar{\ell}_{\mathsf{E'}} + 1)}{\gamma(\mu_{\mathsf{V'E'}} - 2)}, \quad \bar{\ell}_{\mathsf{Z}_2\mathsf{P}} = \frac{2\mu_{\mathsf{V'E'}}(2\gamma\bar{\ell}_{\mathsf{E'}} + 1)}{\gamma(2\mu_{\mathsf{V'E'}} - \phi - 2)}.$$

Proof. A lot of these results can be shown by the mean value identities in Remarks 2.2.15 and 2.2.17. We present here the calculations of $\bar{\ell}_{\mathsf{E}}$ and $\bar{\ell}_{\mathsf{Z}_2\mathsf{E}}$.

We have

$$\gamma_{\mathsf{E}}\bar{\ell}_{\mathsf{E}} = \gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} + \gamma_{\mathsf{E[vert]}}\bar{\ell}_{\mathsf{E[vert]}}.$$

In order to calculate $\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}}$, for each edge e' of $\mathcal{Y}'$ we mark its circumcenter $c(\mathsf{e}')$ with $\mathsf{e}'_o = \mathsf{e}' - c(\mathsf{e}')$ and Φ_γ. Denote by $\widetilde{\mathbb{Q}}'_1$ the grain distribution of E'. Since all the edges of $\Phi_{\mathsf{e}'}$ are translations of e' (see Subsection 2.4.2), we obtain

$$\begin{aligned}
\gamma_{\mathsf{E[hor]}}\bar{\ell}_{\mathsf{E[hor]}} &= \gamma_{\mathsf{E'}} \int\limits_{\mathscr{P}_1^o}\int\limits_{\mathscr{N}_s}\int\limits_{\mathbb{R}^2} \sum_{c\in\varphi_\gamma + c'} \mathbf{1}_{[0,1]^3}(c)\ell(e'_o)\lambda_2(\mathrm{d}c')\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)\widetilde{\mathbb{Q}}'_1(\mathrm{d}e'_o) \\
&= \gamma_{\mathsf{E'}} \int\limits_{\mathscr{P}_1^o} \ell(e'_o) \int\limits_{\mathscr{N}_s} \sum_{(0,0,c_3)\in\varphi_\gamma} \mathbf{1}_{[0,1]}(c_3)\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)\widetilde{\mathbb{Q}}'_1(\mathrm{d}e'_o) \\
&= \gamma\gamma_{\mathsf{E'}} \int\limits_{\mathscr{T}'} \ell(e'_o(T'))\mathbb{Q}_{\mathsf{E'}}(\mathrm{d}T') = \gamma\gamma_{\mathsf{E'}}\bar{\ell}_{\mathsf{E'}} = \frac{1}{2}\gamma\gamma_{\mathsf{V'}}\mu_{\mathsf{V'E'}}\bar{\ell}_{\mathsf{E'}},
\end{aligned}$$

recalling that for a planar tessellation T', $e'_o(T')$ is the edge of T' with circumcenter at the origin o if such an edge exists (otherwise, $e'_o(T') = \emptyset$).

On the other hand, we choose the reference point of a vertical edge of $\widetilde{\mathcal{Y}}$ as its lower endpoint which is also a vertex of $\widetilde{\mathcal{Y}}$. If we mark each vertex $\mathsf{v}' \in \mathsf{V}'$ with Φ_γ then

$$\gamma_{\mathsf{E}[\mathsf{vert}]}\bar{\ell}_{\mathsf{E}[\mathsf{vert}]} = \gamma_{\mathsf{V}'} \int\limits_{\mathscr{N}_s} \int\limits_{\mathbb{R}^2} \sum_{v \in \varphi_\gamma + v'} \mathbf{1}_{[0,1]^3}(v) \ell(e_v[vert]) \lambda_2(\mathrm{d}v') \mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)$$

where $\ell(e_v[vert])$ is the distance between $v \in \varphi_\gamma + v'$ and the upper consecutive point of v, also belonging to $\varphi_\gamma + v'$. Thus

$$\gamma_{\mathsf{E}[\mathsf{vert}]}\bar{\ell}_{\mathsf{E}[\mathsf{vert}]} = \gamma_{\mathsf{V}'} \int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \varphi_\gamma} \mathbf{1}_{[0,1]}(v_3) \ell(e_{(0,0,v_3)}[vert]) \mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma),$$

where $\ell(e_{(0,0,v_3)}[vert])$ is the distance between $(0,0,v_3) \in \varphi_\gamma$ and the upper consecutive point of $(0,0,v_3)$, also belonging to φ_γ. Similar to Equation (35), we get

$$\int\limits_{\mathscr{N}_s} \sum_{(0,0,v_3) \in \varphi_\gamma} \mathbf{1}_{[0,1]}(v_3) \ell(e_{(0,0,v_3)}[vert]) \mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma) = 1.$$

This leads to $\gamma_{\mathsf{E}[\mathsf{vert}]}\bar{\ell}_{\mathsf{E}[\mathsf{vert}]} = \gamma_{\mathsf{V}'}$. Using $\gamma_{\mathsf{E}} = \frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} + \gamma\gamma_{\mathsf{V}'}$ from Proposition 2.4.3, we obtain

$$\bar{\ell}_{\mathsf{E}} = \frac{\gamma_{\mathsf{E}[\mathsf{hor}]}\bar{\ell}_{\mathsf{E}[\mathsf{hor}]} + \gamma_{\mathsf{E}[\mathsf{vert}]}\bar{\ell}_{\mathsf{E}[\mathsf{vert}]}}{\gamma_{\mathsf{E}}} = \frac{\frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} + \gamma_{\mathsf{V}'}}{\frac{1}{2}\gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'} + \gamma\gamma_{\mathsf{V}'}} = \frac{\gamma\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} + 2}{\gamma(\mu_{\mathsf{V}'\mathsf{E}'} + 2)}.$$

Using Remark 2.2.17, we have

$$\gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2\mathsf{E}} = \gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2} + \gamma_{\mathsf{E}[\pi]}\bar{\ell}_{\mathsf{E}[\pi]}.$$

Recall that all horizontal edges of $\widetilde{\mathcal{Y}}$ are not π-edges and a vertical edge of $\widetilde{\mathcal{Y}}$ is a π-edge if its corresponding vertex $\mathsf{v}' \in \mathcal{Y}'$ is a π-vertex. Similarly to the computation of $\gamma_{\mathsf{E}[\mathsf{vert}]}\bar{\ell}_{\mathsf{E}[\mathsf{vert}]}$, we get

$$\gamma_{\mathsf{E}[\pi]}\bar{\ell}_{\mathsf{E}[\pi]} = \gamma_{\mathsf{V}'[\pi]} = \gamma_{\mathsf{V}'}\phi.$$

Furthermore

$$\begin{aligned}\gamma_{\mathsf{Z}_2}\bar{\ell}_{\mathsf{Z}_2} = \gamma_{\mathsf{Z}_2[\mathsf{hor}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{hor}]} + \gamma_{\mathsf{Z}_2[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_2[\mathsf{vert}]} &= 2\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{\ell}_{\mathsf{P}[\mathsf{hor}]} + 2\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{\ell}_{\mathsf{P}[\mathsf{hor}]} + 2\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} \\ &= 4\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{\ell}_{\mathsf{P}[\mathsf{hor}]} + 2\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}.\end{aligned}$$

Since the horizontal plates of $\widetilde{\mathcal{Y}}$ in the column based on a cell z' of $\mathcal{Y}'$ are translations of z', similarly to the calculation of $\gamma_{\mathsf{E}[\mathsf{hor}]}\bar{\ell}_{\mathsf{E}[\mathsf{hor}]}$, we have

$$\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{\ell}_{\mathsf{P}[\mathsf{hor}]} = \gamma\gamma_{\mathsf{Z}'} \int\limits_{\mathscr{T}'} \ell(z'_o(T'))\mathbb{Q}_{\mathsf{Z}'}(\mathrm{d}T') = \gamma\gamma_{\mathsf{Z}'}\bar{\ell}_{\mathsf{Z}'} = 2\gamma\gamma_{\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} = \gamma\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}.$$

Moreover, we mark each vertex $\mathsf{v}' \in \mathsf{V}'$ with Φ_γ and $n_{\mathsf{Z}'}(\mathsf{v}')$. We obtain again the marked point process $\overline{\Psi}$ introduced in the proof of Proposition 2.4.4. We observe that $\mathsf{Z}_1[\mathsf{vert}]$ is a multiset and $\mathsf{Z}_1^{\neq}[\mathsf{vert}] = \mathsf{E}[\mathsf{vert}]$. For any $\mathsf{v}' \in \mathsf{V}'$, we notice that each vertical edge $\mathsf{e}[\mathsf{vert}]$ of the stratum tessellation $\widetilde{\mathcal{Y}}$ which accepts v' as its corresponding vertex in $\mathcal{Y}'$ satisfies that $n_{\mathsf{Z}}(\mathsf{e}[\mathsf{vert}]) = n_{\mathsf{Z}'}(\mathsf{v}')$. Consequently, using mean value identities in [**39**], Theorem 1.1.15, the law of total probability and the independence of Φ_γ and $\mathcal{Y}'$, we find that

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} = \gamma_{\mathsf{E}[\mathsf{vert}]}\mathbb{E}_{\mathsf{E}[\mathsf{vert}]}(\ell(\mathsf{e}[\mathsf{vert}])n_{\mathsf{Z}}(\mathsf{e}[\mathsf{vert}]))$$

$$= \int\limits_{\mathbb{R}^2\times\mathscr{N}_s\times\mathbb{N}} \sum_{(v'_j,\varphi_\gamma,n_j)\in\overline{\psi}} \sum_{v\in\varphi_\gamma+v'_j} \mathbf{1}_{[0,1]^3}(v)\ell(e_v[vert])n_j\,\mathbb{P}_{\overline{\Psi}}(\mathrm{d}\overline{\psi})$$

$$= \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)\int\limits_{\mathscr{N}_s}\int\limits_{\mathbb{R}^2}\sum_{v\in\varphi_\gamma+v'}\mathbf{1}_{[0,1]^3}(v)\ell(e_v[vert])n\,\lambda_2(\mathrm{d}v')\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma).$$

It is shown in the computation of $\gamma_{\mathsf{E}[\mathsf{vert}]}\bar{\ell}_{\mathsf{E}[\mathsf{vert}]}$ that

$$\int\limits_{\mathscr{N}_s}\int\limits_{\mathbb{R}^2}\sum_{v\in\varphi_\gamma+v'}\mathbf{1}_{[0,1]^3}(v)\ell(e_v[vert])\lambda_2(\mathrm{d}v')\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)=1,$$

which implies that

$$\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]} = \gamma_{\mathsf{V}'}\sum_{n=2}^{\infty}n\mathbb{Q}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')=n)$$
$$= \gamma_{\mathsf{V}'}\mathbb{E}_{\mathsf{V}'}(n_{\mathsf{Z}'}(\mathsf{v}')) = \gamma_{\mathsf{V}'}\nu_{\mathsf{V}'\mathsf{Z}'} = \gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-\phi).$$

We arrive at, using $\gamma_{\mathsf{Z}_2} = \gamma\gamma_{\mathsf{V}'}(2\mu_{\mathsf{V}'\mathsf{E}'}-\phi-2)$ from Proposition 2.4.4,

$$\bar{\ell}_{\mathsf{Z}_2\mathsf{E}} = \frac{4\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{\ell}_{\mathsf{P}[\mathsf{hor}]}+2\gamma_{\mathsf{Z}_1[\mathsf{vert}]}\bar{\ell}_{\mathsf{Z}_1[\mathsf{vert}]}+\gamma_{\mathsf{E}[\pi]}\bar{\ell}_{\mathsf{E}[\pi]}}{\gamma_{\mathsf{Z}_2}} = \frac{4\gamma\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}+2\mu_{\mathsf{V}'\mathsf{E}'}-\phi}{\gamma(2\mu_{\mathsf{V}'\mathsf{E}'}-\phi-2)}.$$

□

Theorem 2.4.8. *For the mean areas and mean volumes in* $\widetilde{\mathcal{Y}}$, *we have*

$$\bar{a}_{\mathsf{P}} = \frac{\gamma(\mu_{\mathsf{V}'\mathsf{E}'}-2)\bar{a}_{\mathsf{Z}'}+\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{2\gamma(\mu_{\mathsf{V}'\mathsf{E}'}-1)},\ \bar{a}_{\mathsf{Z}} = 2\left(\bar{a}_{\mathsf{Z}'}+\frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\gamma(\mu_{\mathsf{V}'\mathsf{E}'}-2)}\right),\ \bar{v}_{\mathsf{Z}} = \frac{2}{\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-2)}$$

$$\bar{a}_{\mathsf{Z}_2} = \frac{\gamma(\mu_{\mathsf{V}'\mathsf{E}'}-2)\bar{a}_{\mathsf{Z}'}+\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\gamma(2\mu_{\mathsf{V}'\mathsf{E}'}-\phi-2)},\ \bar{a}_{\mathsf{Z}\mathsf{Z}_2} = 2\left(2\bar{a}_{\mathsf{Z}'}+\frac{\gamma_{\mathsf{Z}_1'^{\neq}}\mathbb{E}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}')m_{\mathsf{Z}'}(\mathsf{s}')\ell(\mathsf{s}'))}{\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'}-2)}\right).$$

Proof. The result for $\bar{v}_{\mathsf{Z}}$ is obvious because $\bar{v}_{\mathsf{Z}} = 1/\gamma_{\mathsf{Z}}$. Using similar argument as for $\bar{\ell}_{\mathsf{E}}$ in Theorem 2.4.7, we can easily determine $\bar{a}_{\mathsf{P}}$ and with the identities $\gamma_{\mathsf{Z}}\bar{a}_{\mathsf{Z}} = \gamma_{\mathsf{Z}_2}\bar{a}_{\mathsf{Z}_2} = 2\gamma_{\mathsf{P}}\bar{a}_{\mathsf{P}}$ we obtain the results for $\bar{a}_{\mathsf{Z}}$ and $\bar{a}_{\mathsf{Z}_2}$. We present here only the computation of $\bar{a}_{\mathsf{Z}\mathsf{Z}_2}$. Recalling that $\mathsf{Z}_2^{\neq}[\mathsf{hor}] = \mathsf{P}[\mathsf{hor}]$ and using the mean value identities in [**39**], we have

$$\gamma_{\mathsf{Z}}\bar{a}_{\mathsf{Z}\mathsf{Z}_2} = \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Big(\sum_{\mathsf{z}_2\in\mathsf{Z}_2:\ \mathsf{z}_2\subset\mathsf{z}} a(\mathsf{z}_2)\Big)$$

$$= \gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Bigg(\sum_{\mathsf{z}_2[\mathsf{hor}]\in\mathsf{Z}_2[\mathsf{hor}]:\ \mathsf{z}_2[\mathsf{hor}]\subset\mathsf{z}} a(\mathsf{z}_2[\mathsf{hor}])\Bigg)+\gamma_{\mathsf{Z}}\mathbb{E}_{\mathsf{Z}}\Bigg(\sum_{\mathsf{z}_2[\mathsf{vert}]\in\mathsf{Z}_2[\mathsf{vert}]:\ \mathsf{z}_2[\mathsf{vert}]\subset\mathsf{z}} a(\mathsf{z}_2[\mathsf{vert}])\Bigg)$$

$$= \gamma_{\mathsf{Z}_2[\mathsf{hor}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{hor}]}\Bigg(a(\mathsf{z}_2[\mathsf{hor}])\Big(\sum_{\mathsf{z}:\ \mathsf{z}_2[\mathsf{hor}]\subset\mathsf{z}}1\Big)\Bigg)+\gamma_{\mathsf{Z}_2[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{vert}]}\Bigg(a(\mathsf{z}_2[\mathsf{vert}])\Big(\sum_{\mathsf{z}:\ \mathsf{z}_2[\mathsf{vert}]\subset\mathsf{z}}1\Big)\Bigg)$$

$$= \gamma_{\mathsf{Z}_2[\mathsf{hor}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{hor}]}(m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{hor}])a(\mathsf{z}_2[\mathsf{hor}]))+\gamma_{\mathsf{Z}_2[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2[\mathsf{vert}]}(m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])a(\mathsf{z}_2[\mathsf{vert}]))$$
$$= \gamma_{\mathsf{P}[\mathsf{hor}]}\mathbb{E}_{\mathsf{P}[\mathsf{hor}]}(n_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}])m_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}])a(\mathsf{p}[\mathsf{hor}]))$$
$$+\gamma_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}(n_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])a(\mathsf{z}_2[\mathsf{vert}])).$$

Since every horizontal plate of $\mathcal{Y}$ is adjacent to 2 cells (that is, $m_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}]) = 2$ for any $\mathsf{p}[\mathsf{hor}] \in \mathsf{P}[\mathsf{hor}]$) and possesses these 2 cells as its owner cells (there are no further owner cells of $\mathsf{p}[\mathsf{hor}]$, that is, $n_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}]) = 2$ for any $\mathsf{p}[\mathsf{hor}] \in \mathsf{P}[\mathsf{hor}]$), we infer that

$$\gamma_{\mathsf{P}[\mathsf{hor}]}\mathbb{E}_{\mathsf{P}[\mathsf{hor}]}(n_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}])m_{\mathsf{Z}}(\mathsf{p}[\mathsf{hor}])a(\mathsf{p}[\mathsf{hor}])) = 4\gamma_{\mathsf{P}[\mathsf{hor}]}\bar{a}_{\mathsf{P}[\mathsf{hor}]} = 4\gamma\gamma_{\mathsf{Z}'}\bar{a}_{\mathsf{Z}'} = 4\gamma.$$

To determine $\gamma_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}(n_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])a(\mathsf{z}_2[\mathsf{vert}]))$, first, we observe that any vertical cell facet $\mathsf{z}_2[\mathsf{vert}] \in \mathsf{Z}_2^{\neq}[\mathsf{vert}]$ has a lower horizontal side which is a translation of a side $\mathsf{s}' \in \mathsf{Z}_1'^{\neq}$. Moreover, $m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}]) = m_{\mathsf{Z}'}(\mathsf{s}')$ and $n_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}]) = n_{\mathsf{Z}'}(\mathsf{s}')$. Hence, for each $\mathsf{s}'_j \in \mathsf{Z}_1'^{\neq}$ we mark $c(\mathsf{s}'_j)$ with $\mathsf{s}'_{jo} = \mathsf{s}'_j - c(\mathsf{s}'_j)$, the point process Φ_γ, the number $\mathsf{K}_j = n_{\mathsf{Z}'}(\mathsf{s}'_j)$ of owner cells of s'_j and the number $\mathsf{T}_j := m_{\mathsf{Z}'}(\mathsf{s}'_j)$ of adjacent cells of s'_j. We obtain a marked point process in the product space $\mathbb{R}^2 \times \mathscr{P}_1^o \times \mathscr{N}_s \times \mathbb{N}^2$, denoted by $\vec{\Psi}$. Denote by $\mathbb{P}_{\vec{\Psi}}$ the distribution of $\vec{\Psi}$. Recall that the reference point of $\mathsf{z}_2[\mathsf{vert}]$ is the midpoint of its lower horizontal side. We have

$$\gamma_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}(n_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])a(\mathsf{z}_2[\mathsf{vert}]))$$
$$= \int \sum_{(c(s'_j),s'_{jo},\varphi_\gamma,k_j,t_j)\in\vec{\psi}} \ \sum_{c\in\varphi_\gamma+c(s'_j)} \mathbf{1}_{[0,1]^3}(c)k_j\cdot t_j\cdot a_c\,\mathbb{P}_{\vec{\Psi}}(\mathrm{d}\vec{\psi})$$
$$= \int\limits_{\mathbb{R}^2\times\mathscr{P}_1^o\times\mathscr{N}_s\times\mathbb{N}^2} \ \sum_{c\in\varphi_\gamma+c'} \mathbf{1}_{[0,1]^3}(c)k\cdot t\cdot a_c\,\Theta_{\vec{\Psi}}(\mathrm{d}(c',s'_o,\varphi_\gamma,k,t)).$$

Here, for $c \in \varphi_\gamma + c'$, a_c is the area of the vertical rectangle whose lower horizontal side possesses c as midpoint and whose upper horizontal side possesses the upper consecutive point of c (also belonging to $\varphi_\gamma + c'$) as midpoint. The notation $\Theta_{\vec{\Psi}}$ denotes the intensity measure of the marked point process $\vec{\Psi}$. Given that with respect to the Palm distribution $\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}$ the side with circumcenter at the origin o has k owner cells and t adjacent cells, the conditional distribution of the first mark s'_o of the side-circumcenter at o is denoted by $\widehat{\mathbb{Q}}^{(k,t)}$. By definition for $B \in \mathcal{B}(\mathbb{R}^2)$ with $0 < \lambda_2(B) < \infty$,

$$\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k, m_{\mathsf{Z}'}(\mathsf{s}') = t) = \frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}}\lambda_2(B)}\mathbb{E}\sum_{\{\mathsf{s}'\in\mathsf{Z}_1'^{\neq}:n_{\mathsf{Z}'}(\mathsf{s}')=k,m_{\mathsf{Z}'}(\mathsf{s}')=t\}} \mathbf{1}_B(c(\mathsf{s}')).$$

Furthermore for $A \in \mathcal{B}(\mathscr{P}_1^o)$,

$$\widehat{\mathbb{Q}}^{(k,t)}(A) = \frac{1}{\mathbb{Q}_{\mathsf{Z}_1'^{\neq}}(n_{\mathsf{Z}'}(\mathsf{s}') = k, m_{\mathsf{Z}'}(\mathsf{s}') = t)}\cdot\frac{1}{\gamma_{\mathsf{Z}_1'^{\neq}}\lambda_2(B)}\times$$
$$\times\mathbb{E}\sum_{\mathsf{s}'\in\mathsf{Z}_1'^{\neq}} \mathbf{1}_B(c(\mathsf{s}'))\mathbf{1}\{n_{\mathsf{Z}'}(\mathsf{s}') = k\}\mathbf{1}\{m_{\mathsf{Z}'}(\mathsf{s}') = t\}\mathbf{1}_A(\mathsf{s}' - c(\mathsf{s}')).$$

Because $1 \leq n_{\mathsf{Z}'}(\mathsf{s}'_j) \leq m_{\mathsf{Z}'}(\mathsf{s}'_j) \leq 2$ for all $\mathsf{s}'_j \in \mathsf{Z}'^{\neq}_1$, using firstly Theorem 1.1.15 for the decomposition of $\Theta_{\vec{\Psi}}$, secondly the law of total probability for the decomposition of the mark distribution of $\vec{\Psi}$ and finally the independence of Φ_γ and $\mathcal{Y}'$, we get

$$\begin{aligned}
&\gamma_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}(n_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])a(\mathsf{z}_2[\mathsf{vert}]))\\
&= \gamma_{\mathsf{Z}'^{\neq}_1}\sum_{k,t=1}^{2}\mathbf{1}\{k \leq t\}\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k, m_{\mathsf{Z}'}(\mathsf{s}') = t)\int\limits_{\mathscr{P}_1^o}\int\limits_{\mathscr{N}_s}\int\limits_{\mathbb{R}^2}\sum_{c\in\varphi_\gamma+c'}\mathbf{1}_{[0,1]^3}(c)k\cdot t\cdot a_c\\
&\qquad\qquad\lambda_2(\mathrm{d}c')\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)\widehat{\mathbb{Q}}^{(k,t)}(\mathrm{d}s'_o)\\
&= \gamma_{\mathsf{Z}'^{\neq}_1}\sum_{k,t=1}^{2}\mathbf{1}\{k \leq t\}\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k, m_{\mathsf{Z}'}(\mathsf{s}') = t)\int\limits_{\mathscr{P}_1^o}\int\limits_{\mathscr{N}_s}\sum_{(0,0,c_3)\in\varphi_\gamma}\mathbf{1}_{[0,1]}(c_3)k\cdot t\cdot a_{(0,0,c_3)}\\
&\qquad\qquad\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)\widehat{\mathbb{Q}}^{(k,t)}(\mathrm{d}s'_o)\\
&= \gamma_{\mathsf{Z}'^{\neq}_1}\sum_{k,t=1}^{2}\mathbf{1}\{k \leq t\}\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}') = k, m_{\mathsf{Z}'}(\mathsf{s}') = t)\int\limits_{\mathscr{P}_1^o}\ell(s'_o)\int\limits_{\mathscr{N}_s}\sum_{(0,0,c_3)\in\varphi_\gamma}\mathbf{1}_{[0,1]}(c_3)k\times t\times\\
&\qquad\qquad\times\ell_{(0,0,c_3)}\mathbb{Q}_\gamma(\mathrm{d}\varphi_\gamma)\widehat{\mathbb{Q}}^{(k,t)}(\mathrm{d}s'_o).
\end{aligned}$$

Here, $\ell_{(0,0,c_3)}$ is the distance from $(0,0,c_3) \in \varphi_\gamma$ to the upper consecutive point of $(0,0,c_3)$ (also belonging to φ_γ). Besides, $a_{(0,0,c_3)}$ is the area of the vertical rectangle whose lower horizontal side possesses $(0,0,c_3) \in \varphi_\gamma$ as midpoint and whose upper horizontal side possesses the upper consecutive point of $(0,0,c_3)$ (also belonging to φ_γ) as midpoint. Similarly to the proof of Theorem 2.2.20, we get

$$\begin{aligned}
&\gamma_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}\mathbb{E}_{\mathsf{Z}_2^{\neq}[\mathsf{vert}]}(n_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])m_{\mathsf{Z}}(\mathsf{z}_2[\mathsf{vert}])a(\mathsf{z}_2[\mathsf{vert}]))\\
&= \gamma_{\mathsf{Z}'^{\neq}_1}\int\limits_{\mathscr{T}'}n_{\mathsf{Z}'}(s'_o(T'))m_{\mathsf{Z}'}(s'_o(T'))\ell(s'_o(T'))\mathbb{Q}_{\mathsf{Z}'^{\neq}_1}(\mathrm{d}T')\\
&= \gamma_{\mathsf{Z}'^{\neq}_1}\mathbb{E}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}')m_{\mathsf{Z}'}(\mathsf{s}')\ell(\mathsf{s}')),
\end{aligned}$$

recalling that for a planar tessellation T', $s'_o(T')$ is the side of T' with circumcenter at the origin o if such a side exists (otherwise, $s'_o(T') = \emptyset$). Consequently

$$\bar{a}_{\mathsf{Z}\mathsf{Z}_2} = \frac{4\gamma + \gamma_{\mathsf{Z}'^{\neq}_1}\mathbb{E}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}')m_{\mathsf{Z}'}(\mathsf{s}')\ell(\mathsf{s}'))}{\gamma\gamma_{\mathsf{Z}'}} = 2\left(2\bar{a}_{\mathsf{Z}'} + \frac{\gamma_{\mathsf{Z}'^{\neq}_1}\mathbb{E}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}')m_{\mathsf{Z}'}(\mathsf{s}')\ell(\mathsf{s}'))}{\gamma\gamma_{\mathsf{V}'}(\mu_{\mathsf{V}'\mathsf{E}'} - 2)}\right).$$

□

Example 2.4.9. If $n_{\mathsf{Z}'}(\mathsf{s}') = 2$ for all $\mathsf{s}' \in \mathsf{Z}'^{\neq}_1$, that is, the stationary random planar tessellation $\mathcal{Y}'$ is side-to-side, then $\mathsf{Z}'^{\neq}_1 = \mathsf{E}'$. We also have $m_{\mathsf{Z}'}(\mathsf{s}') = 2$ for all $\mathsf{s}' \in \mathsf{Z}'^{\neq}_1$. Therefore

$$\gamma_{\mathsf{Z}'^{\neq}_1}\mathbb{E}_{\mathsf{Z}'^{\neq}_1}(n_{\mathsf{Z}'}(\mathsf{s}')m_{\mathsf{Z}'}(\mathsf{s}')\ell(\mathsf{s}')) = 4\gamma_{\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} = 2\gamma_{\mathsf{V}'}\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'} \text{ and } \bar{a}_{\mathsf{Z}\mathsf{Z}_2} = 4\left(\bar{a}_{\mathsf{Z}'} + \frac{\mu_{\mathsf{V}'\mathsf{E}'}\bar{\ell}_{\mathsf{E}'}}{\gamma(\mu_{\mathsf{V}'\mathsf{E}'} - 2)}\right).$$

CHAPTER 3

Marked Poisson hyperplane tessellations

In this chapter, for fixed $t > 0$, a marked Poisson hyperplane process, denoted by Φ_t, is introduced. The results on the lifetime distribution and the dividing-hyperplane distribution of a polytope in the process of marked Poisson hyperplane tessellations generated by $(\Phi_t, t > 0)$ provide us a closer relationship between STIT tessellations and Poisson hyperplane tessellations (Property [**Poisson typical cell**] of STIT tessellations in Page 28 also points to such a relationship).

3.1. The scaling property of Poisson hyperplane tessellations

Let $\mathrm{PHT}(t\Lambda)$ be the stationary Poisson hyperplane tessellation in $\mathbb{R}^d$ generated by the stationary Poisson hyperplane process $\mathrm{PHP}(t\Lambda)$ with intensity measure $t\Lambda$. Here we recall Example 1.1.5 for the definition of Poisson hyperplane processes as well as Example 1.2.4 for that of Poisson hyperplane tessellations.

In the following proposition, we make use of the scaling property of Poisson hyperplane tessellations, which is similar to the scaling property (20) of a STIT tessellation.

Proposition 3.1.1. *The dilated Poisson hyperplane tessellation* $t\,\mathrm{PHT}(t\Lambda)$ *has the same distribution as* $\mathrm{PHT}(\Lambda)$, *i.e.*

$$t\,\mathrm{PHT}(t\Lambda) \overset{\mathcal{D}}{=} \mathrm{PHT}(\Lambda) \quad \text{for all } t > 0. \tag{38}$$

Proof. Consider the mapping $g_1 : A(d, d-1) \to A(d, d-1)$ given by $g_1(H) = tH$ for $H \in A(d, d-1)$. According to the Mapping Theorem in [**9**, Page 18], $g_1(\mathrm{PHP}(t\Lambda)) = t\,\mathrm{PHP}(t\Lambda)$ is a Poisson hyperplane process in $\mathbb{R}^d$. Moreover, the intensity measure of $t\,\mathrm{PHP}(t\Lambda)$, denoted by Θ, is the measure induced from $t\Lambda$ by the function g_1. For any non-negative measurable function f on $A(d, d-1)$, we have, using the decomposition (12) of Λ,

$$\begin{aligned}
\int\limits_{A(d,d-1)} f(H)\Theta(\mathrm{d}H) &= \int\limits_{A(d,d-1)} f(tH)(t\Lambda)(\mathrm{d}H) = t \int\limits_{A(d,d-1)} f(tH)\Lambda(\mathrm{d}H) \\
&= t \int\limits_{G(d,d-1)} \int\limits_{H_0^\perp} f(H_0 + tx))\, \lambda_{H_0^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}H_0) \\
&= t \int\limits_{G(d,d-1)} \int\limits_{H_0^\perp} f(H_0 + x'))\, \lambda_{H_0^\perp}\Big(\mathrm{d}\big(\frac{1}{t}x'\big)\Big)\, \mathbb{Q}(\mathrm{d}H_0)
\end{aligned}$$

$$= \int\limits_{G(d,d-1)} \int\limits_{H_0^\perp} f(H_0 + x')\lambda_{H_0^\perp}(\mathrm{d}x')\,\mathbb{Q}(\mathrm{d}H_0) = \int\limits_{A(d,d-1)} f(H)\Lambda(\mathrm{d}H),$$

which implies that $\Theta = \Lambda$. From the uniqueness theorem for Poisson processes [**30**, Theorem 3.2.1], we infer that $t\,\mathrm{PHP}(t\Lambda) \stackrel{\mathcal{D}}{=} \mathrm{PHP}(\Lambda)$. Consequently,

$$t\,\mathrm{PHT}(t\Lambda) \stackrel{\mathcal{D}}{=} \mathrm{PHT}(\Lambda) \quad \text{for all } t > 0.$$

□

3.2. Construction of marked Poisson hyperplane tessellations

Let Φ be a Poisson process in the product space $A(d, d-1) \times [0, \infty)$ which has intensity measure $\Lambda \otimes \lambda_{[0,\infty)}$, where $\lambda_{[0,\infty)}$ is the Lebesgue measure of $\mathbb{R}$ restricted to $[0, \infty)$. Then, Φ is translation invariant on $A(d, d-1)$ and also translation invariant on $[0, \infty)$. Each point of Φ has the form $(\mathsf{H}, \beta(\mathsf{H}))$ where H is a random hyperplane and $\beta(\mathsf{H})$ is its birth-time.

Now for a fixed time $t > 0$ we put $\Phi_t := \{(\mathsf{H}, \beta(\mathsf{H})) \in \Phi : \beta(\mathsf{H}) \leq t\}$. Consider the mapping $g_2 : A(d, d-1) \times [0, \infty) \to A(d, d-1) \times [0, t]$ given by $g_2(H, x) := (H, x)$ if $x \leq t$ and $\emptyset$ otherwise for $(H, x) \in A(d, d-1) \times [0, \infty)$. According to the Mapping Theorem in [**9**, Page 18], $\Phi_t = g_2(\Phi)$ is a stationary Poisson process in $A(d, d-1) \times [0, t]$ with intensity measure $\Lambda \otimes \lambda_{[0,t]}$, where $\lambda_{[0,t]}$ is the Lebesgue measure of $\mathbb{R}$ restricted to $[0, t]$. From the fact that the hyperplane measure Λ is locally finite, we get

$$(\Lambda \otimes \lambda_{[0,t]})(C \times [0, t]) = t\Lambda(C) < \infty$$

for all compact subsets C of $A(d, d-1)$; see [**30**, Section 13.2] for the topology on $A(d, d-1)$. Hence, according to [**30**, Theorem 3.5.8], Φ_t is independently marked.

Proposition 3.2.1. *Put* $\mathsf{X}_t := \{\mathsf{H} \in A(d, d-1) : (\mathsf{H}, \beta(\mathsf{H})) \in \Phi_t\}$. *Then* X_t *is a stationary Poisson hyperplane process* $\mathrm{PHP}(t\Lambda)$ *with intensity measure* $t\Lambda$*.*

Proof. Indeed, consider the mapping $g_3 : A(d, d-1) \times [0, t] \to A(d, d-1)$ given by $g_3(H, x) = H$ for $(H, x) \in A(d, d-1) \times [0, t]$. Again by the Mapping Theorem in [**9**, Page 18], $\mathsf{X}_t = g_3(\Phi_t)$ is a Poisson process in the set of all $(d-1)$-dimensional affine subspaces of $\mathbb{R}^d$, namely, $A(d, d-1)$.

On the other hand, denoting by Θ_t the intensity measure of X_t, we get, for any $A \in \mathcal{B}(A(d, d-1))$,

$$\Theta_t(A) = (\Lambda \otimes \lambda_{[0,t]})(A \times [0, t]) = t\Lambda(A).$$

Therefore $\Theta_t = t\Lambda$. Then, [**30**, Theorem 3.2.1] gives us the desired statement. □

We conclude that Φ_t is a stationary marked Poisson hyperplane process whose unmarked process is $\mathrm{PHP}(t\Lambda)$. Furthermore, Φ_t generates a marked Poisson hyperplane tessellation in which every k-face p is marked with its $(d-k)$ birth-times, denoted by $\beta_1(\mathsf{p}), \ldots, \beta_{d-k}(\mathsf{p})$. These birth-times are the birth-times of $(d-k)$ hyperplanes whose intersection contains this k-face. We order these random variables in such a way that $0 < \beta_1(\mathsf{p}) < \ldots < \beta_{d-k}(\mathsf{p}) < t$ holds almost surely. Here $k = 0, 1, \ldots, d-1$.

3.3. Some distributions in marked Poisson hyperplane tessellations

3.3.1. Mark distributions of independently marked Poisson hyperplane processes. Because the stationary Poisson process Φ_t in $A(d,d-1)\times[0,t]$ is independently marked (see the beginning of Section 3.2), by definition, the random marks $\{\beta(\mathsf{H}) : (\mathsf{H},\beta(\mathsf{H})) \in \Phi_t\}$ are independently and identically distributed. In this section, our purpose is to compute the distribution $\mathbb{Q}_t$ of the mark $\beta(\mathsf{H})$ which is called the mark distribution of Φ_t. In order to do this, fixed $s \in [0,t]$ and put $\Phi_s := \{(\mathsf{H},\beta(\mathsf{H})) \in \Phi : \beta(\mathsf{H}) \le s\}$. Obviously,

$$\Phi_s = \{(\mathsf{H},\beta(\mathsf{H})) \in \Phi_t : \beta(\mathsf{H}) \le s\}.$$

From the fact that Φ_t is stationary, [**30**, Theorem 3.5.6] shows that, for $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda_d(B) < \infty$,

$$\begin{aligned}
\mathbb{Q}_t([0,s]) &= \frac{\frac{1}{\lambda_d(B)}\mathbb{E}\sum\limits_{(\mathsf{H},\beta(\mathsf{H}))\in\Phi_t}\lambda_{\mathsf{H}}(B)\mathbf{1}_{[0,s]}(\beta(\mathsf{H}))}{\frac{1}{\lambda_d(B)}\mathbb{E}\sum\limits_{(\mathsf{H},\beta(\mathsf{H}))\in\Phi_t}\lambda_{\mathsf{H}}(B)} = \frac{\frac{1}{\lambda_d(B)}\mathbb{E}\sum\limits_{(\mathsf{H},\beta(\mathsf{H}))\in\Phi_s}\lambda_{\mathsf{H}}(B)}{\frac{1}{\lambda_d(B)}\mathbb{E}\sum\limits_{(\mathsf{H},\beta(\mathsf{H}))\in\Phi_t}\lambda_{\mathsf{H}}(B)} \\
&= \frac{\frac{1}{\lambda_d(B)}\mathbb{E}\sum\limits_{\mathsf{H}\in\mathrm{PHP}(s\Lambda)}\lambda_{\mathsf{H}}(B)}{\frac{1}{\lambda_d(B)}\mathbb{E}\sum\limits_{\mathsf{H}\in\mathrm{PHP}(t\Lambda)}\lambda_{\mathsf{H}}(B)} = \frac{s}{t},
\end{aligned}$$

according to [**30**, Theorem 4.4.3]. Here λ_H denotes the $(d-1)$-dimensional Lebesgue measure on a $(d-1)$-dimensional affine subspace H of $\mathbb{R}^d$. Thus, the mark $\beta(\mathsf{H})$ is uniformly distributed on $[0,t]$.

3.3.2. Lifetime distributions of polytopes in marked Poisson hyperplane tessellations. Let $(\mathsf{p},\beta(\mathsf{p}))$ be a random k-face with $k \in \{0,1,\dots,d-1\}$ or a random cell marked with its birth-time (the time that p appears) in the process of marked Poisson hyperplane tessellations generated by $(\Phi_t, t>0)$. Note that in the case that p is a random k-face, $\beta(\mathsf{p})$ must not be $\beta_{d-k}(\mathsf{p})$. Recall that $\langle p\rangle$ is the set of all hyperplanes which intersect some polytope p. The following proposition shows us the conditional lifetime distribution of p.

Proposition 3.3.1. *The conditional lifetime of the random polytope* p *given a realization* p *of* p *is exponentially distributed with parameter* $\Lambda(\langle p\rangle)$.

Proof. The conditional lifetime of the random polytope p given a realization p of p is the lifetime of p denoted by $\tau(p)$. Fix $s>0$. We have

$$\mathbb{P}(\tau(p)>s) = \mathbb{P}(\Phi(\langle p\rangle\times[\beta(p),\beta(p)+s])=0) = e^{-(\Lambda\otimes\lambda_{[0,\infty)})(\langle p\rangle\times[\beta(p),\beta(p)+s])} = e^{-s\Lambda(\langle p\rangle)}$$

and the assertion follows. $\square$

3.3.3. Dividing-hyperplane distributions of polytopes in marked Poisson hyperplane tessellations. Let $(\mathsf{p},\beta(\mathsf{p}))$ be a random k-face with $k \in \{0,1,\dots,d-1\}$ or a random cell marked with its birth-time in the process of marked Poisson hyperplane tessellations generated by $(\Phi_t, t>0)$. Further let p be a realization of p. Among all $(\mathsf{H},\beta(\mathsf{H})) \in \Phi$ satisfying $\mathsf{H} \in \langle p\rangle$, there is a random hyperplane

H having the smallest birth-time $\beta(\mathsf{H})$. This uniquely determined random hyperplane is denoted by $h(p)$ and called the dividing-hyperplane of p. The definition of dividing-hyperplanes coincides with the one used only for cells in the construction of STIT tessellations in Subsection 1.4.2.

Proposition 3.3.2. *The conditional distribution of the dividing-hyperplane of the random polytope* p *given a realization* p *of* p *is* $\Lambda(\cdot \cap \langle p \rangle)/\Lambda(\langle p \rangle)$.

The conditional distribution of the dividing-hyperplane of the random polytope p given a realization p of p is the distribution of $h(p)$. If $\beta := \beta(h(p))$ denotes the birth-time of $h(p)$, we find that, for $A \in \mathcal{B}(A(d, d-1))$,

$$
\begin{aligned}
&\mathbb{P}(h(p) \in A) \\
&= \mathbb{P}(\mathsf{H} \in A | \text{ there is only 1 hyperplane, denoted by } \mathsf{H}, \text{ of } \mathsf{X}_\beta \text{ which belongs to } \langle p \rangle) \\
&= \left[\mathbb{P}(\text{there is only 1 hyperplane of } \mathsf{X}_\beta \text{ which belongs to } \langle p \rangle)\right]^{-1} \times \\
&\qquad \times \mathbb{P}(\text{there is only 1 hyperplane of } \mathsf{X}_\beta \text{ which belongs to } A \cap \langle p \rangle \text{ and} \\
&\qquad\qquad \text{there is no hyperplane of } \mathsf{X}_\beta \text{ belonging to } \langle p \rangle \backslash A) \\
&= \frac{\mathbb{P}(\mathsf{X}_\beta(A \cap \langle p \rangle) = 1, \mathsf{X}_\beta(\langle p \rangle \setminus A) = 0)}{\mathbb{P}(\mathsf{X}_\beta(\langle p \rangle) = 1)} = \frac{\mathbb{P}(\mathsf{X}_\beta(A \cap \langle p \rangle) = 1) \cdot \mathbb{P}(\mathsf{X}_\beta(\langle p \rangle \setminus A) = 0)}{\mathbb{P}(\mathsf{X}_\beta(\langle p \rangle) = 1)} \\
&= \frac{\beta \Lambda(A \cap \langle p \rangle) e^{-\beta \Lambda(A \cap \langle p \rangle)} e^{-\beta \Lambda(\langle p \rangle \setminus A)}}{\beta \Lambda(\langle p \rangle) e^{-\beta \Lambda(\langle p \rangle)}} = \frac{\Lambda(A \cap \langle p \rangle) e^{-\beta \Lambda(\langle p \rangle)}}{\Lambda(\langle p \rangle) e^{-\beta \Lambda(\langle p \rangle)}} = \frac{\Lambda(A \cap \langle p \rangle)}{\Lambda(\langle p \rangle)}
\end{aligned}
$$

and we complete the proof.

3.4. The birth-time vector marked V_j-weighted typical k-face

At first we define a V_j-weighted typical k-dimensional face of the stationary Poisson hyperplane tessellation $\mathrm{PHT}(t\Lambda)$. For $k \in \{0, \dots, d-1\}$ we denote by $\mathscr{F}_k^{(t)}$ the process of k-dimensional faces of $\mathrm{PHT}(t\Lambda)$.

Definition 3.4.1. Fix $d \geq 2$, $k \in \{0, \dots, d-1\}$ and $j \in \{0, \dots, k\}$. We introduce a probability measure $\mathbb{P}_{k,j}^{\mathrm{PHT}(t\Lambda)}$ on $\mathscr{P}_k^o$ as follows:

$$
\mathbb{P}_{k,j}^{\mathrm{PHT}(t\Lambda)}(A) := \frac{\mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{F}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, \mathbf{1}_A(\mathsf{p} - c(\mathsf{p}))\, V_j(\mathsf{p})}{\mathbb{E} \sum\limits_{\mathsf{p} \in \mathscr{F}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_j(\mathsf{p})},
$$

where $A \in \mathcal{B}(\mathscr{P}_k^o)$ and $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$.

A random polytope with distribution $\mathbb{P}_{k,j}^{\mathrm{PHT}(t\Lambda)}$ is called a *V_j-weighted typical k-dimensional face* of $\mathrm{PHT}(t\Lambda)$ and will henceforth be denoted by $\mathsf{F}_{k,j}^{(t)}$. Sometimes we write $\mathbb{P}_{\mathsf{F}_{k,j}^{(t)}}$ instead of $\mathbb{P}_{k,j}^{\mathrm{PHT}(t\Lambda)}$ with the same meaning.

Proposition 3.4.2. *Let $d \geq 2$, $k \in \{0,\dots,d-1\}$, $i,j \in \{0,\dots,k\}$ and $f : \mathscr{P}_k^o \to \mathbb{R}$ be non-negative and measurable. A relationship between $\mathsf{F}_{k,i}^{(t)}$ and $\mathsf{F}_{k,j}^{(t)}$ is*

$$\mathbb{E}f(\mathsf{F}_{k,i}^{(t)}) = \frac{\mathbb{E}V_j(\mathsf{F}_{k,0}^{(t)})}{\mathbb{E}V_i(\mathsf{F}_{k,0}^{(t)})}\mathbb{E}[f(\mathsf{F}_{k,j}^{(t)})V_i(\mathsf{F}_{k,j}^{(t)})V_j(\mathsf{F}_{k,j}^{(t)})^{-1}].$$

Proof. Similar to Corollary 1.4.9. □

Definition 3.4.3. For each k-dimensional face $\mathsf{p} \in \mathscr{F}_k^{(t)}$ we mark $c(\mathsf{p})$ with $\mathsf{p}_o := \mathsf{p} - c(\mathsf{p})$ and the vector of $(d-k)$ birth-times of p_o, namely, $(\beta_1(\mathsf{p}_o),\dots,\beta_{d-k}(\mathsf{p}_o)) = (\beta_1(\mathsf{p}),\dots,\beta_{d-k}(\mathsf{p}))$. This give rise to a marked point process $\widetilde{\mathscr{F}}_k^{(t)}$ in $\mathbb{R}^d \times \mathscr{P}_k^o \times \Delta(t)$. Now we introduce a probability measure $\widetilde{\mathbb{P}}_{k,j}^{\mathrm{PHT}(t\Lambda)}$ on $\mathscr{P}_k^o \times \Delta(t)$ as follows:

$$\widetilde{\mathbb{P}}_{k,j}^{\mathrm{PHT}(t\Lambda)}[A \times ((B_1 \times \ldots \times B_{d-k}) \cap \Delta(t))] = \left[\mathbb{E}\sum_{\mathsf{p}\in\mathscr{F}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\,V_j(\mathsf{p})\right]^{-1} \times$$

$$\mathbb{E}\sum_{(c(\mathsf{p}),\mathsf{p}_o,\beta_1(\mathsf{p}_o),\dots,\beta_{d-k}(\mathsf{p}_o))\in\widetilde{\mathscr{F}}_k^{(t)},\,c(\mathsf{p})\in B} \mathbf{1}_A(\mathsf{p}_o)V_j(\mathsf{p}_o)\mathbf{1}\{0<\beta_1(\mathsf{p}_o)<\ldots<\beta_{d-k}(\mathsf{p}_o)<t\}\times$$

$$\times\mathbf{1}_{B_1}(\beta_1(\mathsf{p}_o))\ldots\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{p}_o))$$

where $A \in \mathcal{B}(\mathscr{P}_k^o)$, $B \in \mathcal{B}$ with $0 < \lambda_d(B) < \infty$ and $B_1,\dots,B_{d-k} \in \mathcal{B}((0,t))$. Recall that $\Delta(t)$ is a $(d-k)$-simplex (a subset of $\mathbb{R}^{d-k}$) defined as

$$\Delta(t) = \{(r_1,\dots,r_{d-k}) \in \mathbb{R}^{d-k} : 0 < r_1 < \ldots < r_{d-k} < t\}.$$

A vector of a random polytope and $(d-k)$ random times with distribution $\widetilde{\mathbb{P}}_{k,j}^{\mathrm{PHT}(t\Lambda)}$ is called a *birth-time-vector marked V_j-weighted typical k-dimensional face* of $\mathrm{PHT}(t\Lambda)$ and will be henceforth denoted by $(\mathsf{F}_{k,j}^{(t)},\beta_1(\mathsf{F}_{k,j}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,j}^{(t)}))$. Hence, sometimes we use the notation $\mathbb{P}_{\mathsf{F}_{k,j}^{(t)},\beta_1(\mathsf{F}_{k,j}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,j}^{(t)})}$ instead of $\widetilde{\mathbb{P}}_{k,j}^{\mathrm{PHT}(t\Lambda)}$.

Note that $\mathbb{P}_{\mathsf{F}_{k,j}^{(t)},\beta_1(\mathsf{F}_{k,j}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,j}^{(t)})}(A \times \Delta(t))) = \mathbb{P}_{k,j}^{\mathrm{PHT}(t\Lambda)}(A) = \mathbb{P}_{\mathsf{F}_{k,j}^{(t)}}(A)$. The next proposition is a version of Proposition 1.4.15 for Poisson hyperplane tessellations.

Proposition 3.4.4. *Let $d \geq 2$, $k \in \{0,\dots,d-1\}$, $i,j \in \{0,\dots,k\}$ and $f : \mathscr{P}_k^o \times (0,t)^{d-k} \to \mathbb{R}$ be non-negative and measurable. A relationship between $(\mathsf{F}_{k,i}^{(t)},\beta_1(\mathsf{F}_{k,i}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,i}^{(t)}))$ and $(\mathsf{F}_{k,j}^{(t)},\beta_1(\mathsf{F}_{k,j}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,j}^{(t)}))$ is given by*

$$\mathbb{E}[f(\mathsf{F}_{k,i}^{(t)},\beta_1(\mathsf{F}_{k,i}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,i}^{(t)}))\mathbf{1}_{\Delta(t)}((\beta_1(\mathsf{F}_{k,i}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,i}^{(t)})))] = \frac{\mathbb{E}V_j(\mathsf{F}_{k,0}^{(t)})}{\mathbb{E}V_i(\mathsf{F}_{k,0}^{(t)})}\times$$

$$\times\mathbb{E}[f(\mathsf{F}_{k,j}^{(t)},\beta_1(\mathsf{F}_{k,j}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,j}^{(t)}))\mathbf{1}_{\Delta(t)}((\beta_1(\mathsf{F}_{k,j}^{(t)}),\dots,\beta_{d-k}(\mathsf{F}_{k,j}^{(t)})))V_i(\mathsf{F}_{k,j}^{(t)})V_j(\mathsf{F}_{k,j}^{(t)})^{-1}].$$

3.5. The independence between the V_j-weighted typical k-face and its birth-time vector

Theorem 3.5.1. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$ and $j \in \{0, \dots, k\}$. The V_j-weighted typical k-dimensional face $\mathsf{F}_{k,j}^{(t)}$ of the stationary Poisson hyperplane tessellation* $\mathrm{PHT}(t\Lambda)$ *is independent of its birth-time vector $(\beta_1(\mathsf{F}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{F}_{k,j}^{(t)}))$.*

Proof. For each k-face $\mathsf{p} \in \mathscr{F}_k^{(t)}$ of $\mathrm{PHT}(t\Lambda)$, we mark its circumcenter $c(\mathsf{p})$ with $\mathsf{p}_o = \mathsf{p} - c(\mathsf{p})$ and the birth-time vector $(\beta_1(\mathsf{p}_o), \dots, \beta_{d-k}(\mathsf{p}_o))$ or $(\beta_1', \dots, \beta_{d-k}')$ for brevity. We obtain the marked point process $\widetilde{\mathscr{F}}_k^{(t)}$; see Definition 3.4.3. Recall that Φ_t is the corresponding marked Poisson hyperplane process of $\mathrm{PHT}(t\Lambda)$; see Section 3.2. Sometimes we use the notation $\overline{\Phi_t}$ instead of $\widetilde{\mathscr{F}}_k^{(t)}$ to emphasize the connection between Φ_t and $\widetilde{\mathscr{F}}_k^{(t)}$. For $A \in \mathcal{B}(\mathscr{P}_k^o)$, $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda_d(B) < \infty$ and $B_j \in \mathcal{B}([0,t])$, $j = 1, 2, \dots, d-k$, Definition 3.4.3 gives us

$$\mathbb{P}_{\mathsf{F}_{k,j}^{(t)}, \beta_1(\mathsf{F}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{F}_{k,j}^{(t)})}[A \times ((B_1 \times \dots \times B_{d-k}) \cap \Delta(t))] = \Big[\mathbb{E} \sum_{\mathsf{p} \in \mathscr{F}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p})) V_j(\mathsf{p})\Big]^{-1} \times$$

$$\times \mathbb{E} \sum_{(c(\mathsf{p}), \mathsf{p}_o, \beta_1', \dots, \beta_{d-k}') \in \overline{\Phi_t},\, c(\mathsf{p}) \in B} \mathbf{1}_A(\mathsf{p}_o) V_j(\mathsf{p}_o) \mathbf{1}_{\Delta(t)}((\beta_1', \dots, \beta_{d-k}')) \mathbf{1}_{B_1}(\beta_1') \dots \mathbf{1}_{B_{d-k}}(\beta_{d-k}').$$

We observe that

$$\mathbb{E} \sum_{(c(\mathsf{p}), \mathsf{p}_o, \beta_1', \dots, \beta_{d-k}') \in \overline{\Phi_t}} \mathbf{1}_B(c(\mathsf{p})) \mathbf{1}_A(\mathsf{p}_o) V_j(\mathsf{p}_o) \mathbf{1}_{\Delta(t)}((\beta_1', \dots, \beta_{d-k}')) \mathbf{1}_{B_1}(\beta_1') \dots \mathbf{1}_{B_{d-k}}(\beta_{d-k}')$$

$$= \mathbb{E} \sum_{((\mathsf{H}_1, \beta_1), \dots, (\mathsf{H}_{d-k}, \beta_{d-k})) \in (\Phi_{t,\neq})^{d-k}} \; \sum_{(c(\mathsf{p}), \mathsf{p}_o, \beta_1', \dots, \beta_{d-k}') \in \overline{\Phi_t} : c(\mathsf{p}) + \mathsf{p}_o \subset \mathsf{H}_1 \cap \dots \cap \mathsf{H}_{d-k}} \mathbf{1}_B(c(\mathsf{p})) \mathbf{1}_A(\mathsf{p}_o) \times$$

$$\times V_j(\mathsf{p}_o) \mathbf{1}_{\Delta(t)}((\beta_1, \dots, \beta_{d-k})) \mathbf{1}_{B_1}(\beta_1) \dots \mathbf{1}_{B_{d-k}}(\beta_{d-k}).$$

Here $(\Phi_{t,\neq})^{d-k} := (\Phi_t)^{d-k} \llcorner (A(d,d-1) \times [0,t])^{d-k}_{\neq}$, where

$$(A(d,d-1) \times [0,t])^{d-k}_{\neq} := \{((H_1, t_1), \dots, (H_{d-k}, t_{d-k})) \in (A(d,d-1) \times [0,t])^{d-k} :$$
$$(H_j, t_j) \text{ pairwise distinct}\} \text{ and}$$

$(\Phi_t)^{d-k} \llcorner (A(d,d-1) \times [0,t])^{d-k}_{\neq}$ is the restriction of $(\Phi_t)^{d-k}$ to $(A(d,d-1) \times [0,t])^{d-k}_{\neq}$ given by $\big((\Phi_t)^{d-k} \llcorner (A(d,d-1) \times [0,t])^{d-k}_{\neq}\big)(\widetilde{B}) := (\Phi_t)^{d-k}(\widetilde{B} \cap (A(d,d-1) \times [0,t])^{d-k}_{\neq})$ for all $\widetilde{B} \in \mathcal{B}((A(d,d-1) \times [0,t])^{d-k})$.

Note that if $((\mathsf{H}_1, \beta_1), \dots, (\mathsf{H}_{d-k}, \beta_{d-k})) \in (\Phi_{t,\neq})^{d-k}$ is already chosen, the set

$$\{(c(\mathsf{p}), \mathsf{p}_o, \beta_1', \dots, \beta_{d-k}') \in \overline{\Phi_t} : c(\mathsf{p}) + \mathsf{p}_o \subset \mathsf{H}_1 \cap \dots \cap \mathsf{H}_{d-k}\}$$

is equal to the set $\{(c(\mathsf{p}), \mathsf{p}_o, \beta_1', \dots, \beta_{d-k}') \in \overline{\Phi_t} : \beta_1' = \beta_1, \dots, \beta_{d-k}' = \beta_{d-k}\}$. By Slivnyak-Mecke formula [**30**, Corollary 3.2.3], we have

$$\mathbb{E} \sum_{(c(\mathsf{p}), \mathsf{p}_o, \beta_1', \dots, \beta_{d-k}') \in \overline{\Phi_t}} \mathbf{1}_B(c(\mathsf{p})) \mathbf{1}_A(\mathsf{p}_o) V_j(\mathsf{p}_o) \mathbf{1}_{\Delta(t)}((\beta_1', \dots, \beta_{d-k}')) \mathbf{1}_{B_1}(\beta_1') \dots \mathbf{1}_{B_{d-k}}(\beta_{d-k}')$$

$$= \int\limits_{A(d,d-1)\times[0,t]} \dots \int\limits_{A(d,d-1)\times[0,t]} \mathbb{E} \sum_{\overline{(c(\mathsf{p}),\mathsf{p}_o,\beta'_1,\dots,\beta'_{d-k})\in\Phi_t+\sum\limits_{j=1}^{d-k}\delta_{(H_j,\beta_j)}}:c(\mathsf{p})+\mathsf{p}_o\subset H_1\cap\dots\cap H_{d-k},c(\mathsf{p})\in B} \mathbf{1}_A(\mathsf{p}_o)\times$$

$$\times V_j(\mathsf{p}_o)\mathbf{1}_{\Delta(t)}((\beta_1,\dots,\beta_{d-k}))\mathbf{1}_{B_1}(\beta_1)\dots\mathbf{1}_{B_{d-k}}(\beta_{d-k})\Theta'_t(\mathrm{d}(H_1,\beta_1))\dots\Theta'_t(\mathrm{d}(H_{d-k},\beta_{d-k})),$$

where Θ'_t is the intensity measure of the marked Poisson hyperplane process Φ_t. In particular, $\Theta'_t = \Lambda\otimes\lambda_{[0,t]}$; see Section 3.2. Note that $\overline{\mathrm{PHP}(t\Lambda)\cup H_1}\cup\dots\cup H_{d-k}$ is the unmarked process of $\Phi_t + \sum\limits_{j=1}^{d-k}\delta_{(H_j,\beta_j)}$. Moreover $\Phi_t + \sum\limits_{j=1}^{d-k}\delta_{(H_j,\beta_j)}$ is the marked point process constructed in the following way: Each circumcenter $c(\mathsf{p})$ of a k-face p of the Poisson hyperplane tessellation generated by $\mathrm{PHP}(t\Lambda)\cup H_1\cup\dots\cup H_{d-k}$ is marked with $\mathsf{p}_o = \mathsf{p} - c(\mathsf{p})$ and the vector of $(d-k)$ birth-times $(\beta'_1,\dots,\beta'_{d-k}) = (\beta_1(\mathsf{p}),\dots,\beta_{d-k}(\mathsf{p}))$. Recall that the components of $(\beta_1(\mathsf{p}),\dots,\beta_{d-k}(\mathsf{p}))$ are the birth-times of $(d-k)$ hyperplanes in $\mathrm{PHP}(t\Lambda)\cup H_1\cup\dots\cup H_{d-k}$ whose intersection contains p.

The Poisson hyperplane process $\mathrm{PHP}(t\Lambda)\cup H_1\cup\dots\cup H_{d-k}$ generates a Poisson hyperplane process of dimension $(k-1)$ in the k-dimensional affine subspace $H_1\cap\dots\cap H_{d-k}$ of $\mathbb{R}^d$. The corresponding k-dimensional Poisson hyperplane tessellation in $H_1\cap\dots\cap H_{d-k}$ is denoted by T_k. For each cell of T_k, we mark its circumcenter with the shifted cell whose circumcenter is at the origin o and denote by Ψ the corresponding marked point process in $(H_1\cap\dots\cap H_{d-k})\times\mathscr{P}_k^o$. Let $Z(\mathsf{T}_k)$ be the set of cells of T_k. Further, let $\gamma^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\dots\cap H_{d-k}}$ and $\mathbb{Q}^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\dots\cap H_{d-k}}$ be the intensity and the grain distribution of $Z(\mathsf{T}_k)$, respectively. According to Corollary 1.1.16, $\mathbb{Q}^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\dots\cap H_{d-k}}$ equals the mark distribution of Ψ. Denote by $\lambda_{H_1\cap\dots\cap H_{d-k}}$ the k-dimensional Lebesgue measure on the k-dimensional affine subspace $H_1\cap\dots\cap H_{d-k}$ of $\mathbb{R}^d$. We get, using Theorem 1.1.11(a),

$$\mathbb{E}\sum_{(c(\mathsf{p}),\mathsf{p}_o,\beta'_1,\dots,\beta'_{d-k})\in\overline{\Phi_t}} \mathbf{1}_B(c(\mathsf{p}))\mathbf{1}_A(\mathsf{p}_o)V_j(\mathsf{p}_o)\mathbf{1}_{\Delta(t)}((\beta'_1,\dots,\beta'_{d-k}))\mathbf{1}_{B_1}(\beta'_1)\dots\mathbf{1}_{B_{d-k}}(\beta'_{d-k})$$

$$= \int\limits_{A(d,d-1)\times[0,t]} \dots \int\limits_{A(d,d-1)\times[0,t]} \mathbf{1}_{\Delta(t)}((\beta_1,\dots,\beta_{d-k}))\mathbf{1}_{B_1}(\beta_1)\dots\mathbf{1}_{B_{d-k}}(\beta_{d-k})\times$$

$$\times\mathbb{E}\sum_{(c(\mathsf{p}),\mathsf{p}_o)\in\Psi}\mathbf{1}_B(c(\mathsf{p}))\mathbf{1}_A(\mathsf{p}_o)V_j(\mathsf{p}_o)\Theta'_t(\mathrm{d}(H_1,\beta_1))\dots\Theta'_t(\mathrm{d}(H_{d-k},\beta_{d-k}))$$

$$= \int\limits_{A(d,d-1)\times B_{d-k}} \dots \int\limits_{A(d,d-1)\times B_1} \mathbf{1}_{\Delta(t)}((\beta_1,\dots,\beta_{d-k}))\lambda_{H_1\cap\dots\cap H_{d-k}}(B)\gamma^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\dots\cap H_{d-k}}$$

$$\int\limits_{\mathscr{P}_k^o}\mathbf{1}_A(p_o)V_j(p_o)\mathbb{Q}^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\dots\cap H_{d-k}}(\mathrm{d}p_o)(\Lambda\otimes\lambda_{[0,t]})(\mathrm{d}(H_1,\beta_1))\dots(\Lambda\otimes\lambda_{[0,t]})(\mathrm{d}(H_{d-k},\beta_{d-k}))$$

$$= \int\limits_{B_{d-k}}\dots\int\limits_{B_1}\mathbf{1}_{\Delta(t)}((\beta_1,\dots,\beta_{d-k}))\lambda_{[0,t]}(\mathrm{d}\beta_1)\dots\lambda_{[0,t]}(\mathrm{d}\beta_{d-k})\times\int\limits_{A(d,d-1)}\dots\int\limits_{A(d,d-1)}$$

$$\lambda_{H_1\cap\ldots\cap H_{d-k}}(B)\gamma^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\ldots\cap H_{d-k}}\int\limits_A V_j(p_o)\mathbb{Q}^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\ldots\cap H_{d-k}}(\mathrm{d}p_o)\Lambda(\mathrm{d}H_1)\ldots\Lambda(\mathrm{d}H_{d-k}).$$

Therefore, if λ_{d-k} denotes the Lebesgue measure on $\mathbb{R}^{d-k}$,

$$\mathbb{P}_{\mathsf{F}^{(t)}_{k,j},\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})}[A\times((B_1\times\ldots\times B_{d-k})\cap\Delta(t))] =$$

$$= \left[\mathbb{E}\sum_{\mathsf{p}\in\mathscr{F}^{(t)}_k}\mathbf{1}_B(c(\mathsf{p}))V_j(\mathsf{p})\right]^{-1}\lambda_{d-k}((B_1\times\ldots\times B_{d-k})\cap\Delta(t))\int\limits_{A(d,d-1)}\ldots\int\limits_{A(d,d-1)}$$

$$\lambda_{H_1\cap\ldots\cap H_{d-k}}(B)\gamma^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\ldots\cap H_{d-k}}\int\limits_A V_j(p_o)\mathbb{Q}^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\ldots\cap H_{d-k}}(\mathrm{d}p_o)\Lambda(\mathrm{d}H_1)\ldots\Lambda(\mathrm{d}H_{d-k}), \quad (39)$$

which implies that $\mathsf{F}^{(t)}_{k,j}$ is independent of $(\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j}))$. □

Corollary 3.5.2. *Let $d\geq 2$, $k\in\{0,\ldots,d-1\}$ and $j\in\{0,\ldots,k\}$. The joint distribution of the birth-times $\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})$ of the V_j-weighted typical k-dimensional face $\mathsf{F}^{(t)}_{k,j}$ of the stationary Poisson hyperplane tessellation $\mathrm{PHT}(t\Lambda)$ is the uniform distribution on the simplex $\Delta(t)$.*

Proof. For $A\in\mathcal{B}(\mathscr{P}^o_k)$, Equation (39) gives us

$$\mathbb{P}_{\mathsf{F}^{(t)}_{k,j}}(A) = \mathbb{P}_{\mathsf{F}^{(t)}_{k,j},\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})}(A\times\Delta(t))$$

$$= \left[\mathbb{E}\sum_{\mathsf{p}\in\mathscr{F}^{(t)}_k}\mathbf{1}_B(c(\mathsf{p}))V_j(\mathsf{p})\right]^{-1}\lambda_{d-k}(\Delta(t))\int\limits_{A(d,d-1)}\ldots\int\limits_{A(d,d-1)}\lambda_{H_1\cap\ldots\cap H_{d-k}}(B)\times$$

$$\times\gamma^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\ldots\cap H_{d-k}}\int\limits_A V_j(p_o)\mathbb{Q}^{\mathrm{PHP}(t\Lambda)}_{H_1\cap\ldots\cap H_{d-k}}(\mathrm{d}p_o)\Lambda(\mathrm{d}H_1)\ldots\Lambda(\mathrm{d}H_{d-k}).$$

Hence

$$\mathbb{P}_{\mathsf{F}^{(t)}_{k,j},\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})}(A\times((B_1\times\ldots\times B_{d-k})\cap\Delta(t)))$$
$$= \mathbb{P}_{\mathsf{F}^{(t)}_{k,j}}(A)\cdot\frac{\lambda_{d-k}((B_1\times\ldots\times B_{d-k})\cap\Delta(t))}{\lambda_{d-k}(\Delta(t))}.$$

On the other hand, using the independence of $\mathsf{F}^{(t)}_{k,j}$ and its birth-time vector $(\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j}))$ shown in Theorem 3.5.1, we get, for $B_1,\ldots,B_{d-k}\in\mathcal{B}((0,t))$,

$$\mathbb{P}_{\mathsf{F}^{(t)}_{k,j},\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})}(A\times((B_1\times\ldots\times B_{d-k})\cap\Delta(t)))$$
$$= \mathbb{P}_{\mathsf{F}^{(t)}_{k,j}}(A)\cdot\mathbb{P}_{\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})}((B_1\times\ldots\times B_{d-k})\cap\Delta(t)).$$

We arrive at

$$\mathbb{P}_{\beta_1(\mathsf{F}^{(t)}_{k,j}),\ldots,\beta_{d-k}(\mathsf{F}^{(t)}_{k,j})}((B_1\times\ldots\times B_{d-k})\cap\Delta(t)) = \frac{\lambda_{d-k}((B_1\times\ldots\times B_{d-k})\cap\Delta(t))}{\lambda_{d-k}(\Delta(t))},$$

which completes our proof. □

3.6. Relationships between STIT tessellations and Poisson hyperplane tessellations

In our arguments below, we need two identities describing the distributions of $\mathsf{MP}_{k,j}^{(t)}$ and $(\mathsf{MP}_{k,j}^{(t)}, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))$; see Definitions 1.4.6 and 1.4.10, in terms of weighted faces in Poisson hyperplane tessellations. In [**31**, Theorem 3 and Corollary 3] these fundamental connections between STIT tessellations and Poisson hyperplane tessellations have been established for $j = 0$, namely:

$$\mathbb{P}_{k,0}^{(t)} = \int_0^t \frac{ds^{d-1}}{t^d} \mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)} \mathrm{d}s \tag{40}$$

$$\widehat{\mathbb{P}}_{k,0}^{(t)} = \int_0^t \frac{ds^{d-1}}{t^d} \left[\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)} \otimes \delta_s\right] \mathrm{d}s, \tag{41}$$

where for $s \in (0,t)$, the Dirac measure δ_s is defined by

$$\delta_s(B) = \begin{cases} 1, & \text{if } s \in B, \\ 0, & \text{if } s \notin B, \end{cases}$$

for $B \in \mathcal{B}((0,t))$. Equations (40) and (41) can be rewritten in integral form as follows:

$$\int_{\mathscr{P}_k^o} f(p)\mathbb{P}_{k,0}^{(t)}(\mathrm{d}p) = \int_0^t \frac{ds^{d-1}}{t^d} \int_{\mathscr{P}_k^o} f(p)\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)}(\mathrm{d}p)\mathrm{d}s \tag{42}$$

and

$$\begin{aligned} \int_{\mathscr{P}_k^o \times (0,t)} g(p,u)\widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,u)) &= \int_0^t \frac{ds^{d-1}}{t^d} \int_0^t \int_{\mathscr{P}_k^o} g(p,u)\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)}(\mathrm{d}p)\delta_s(\mathrm{d}u)\mathrm{d}s \\ &= \int_0^t \frac{ds^{d-1}}{t^d} \int_{\mathscr{P}_k^o} g(p,s)\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)}(\mathrm{d}p)\mathrm{d}s \end{aligned}$$

for all non-negative measurable functions $f : \mathscr{P}_o^k \to \mathbb{R}$ and $g : \mathscr{P}_o^k \times (0,t) \to \mathbb{R}$. In particular, if g has the form $g(p,u) = f(p)\mathbf{1}_B(u)$ where $B \in \mathcal{B}((0,t))$ then

$$\int_{\mathscr{P}_k^o \times (0,t)} f(p)\mathbf{1}_B(u)\widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,u)) = \int_0^t \frac{ds^{d-1}}{t^d} \int_{\mathscr{P}_k^o} f(p)\mathbf{1}_B(s)\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)}(\mathrm{d}p)\mathrm{d}s,$$

It leads to
$$\int_{\mathscr{P}_k^o \times (0,t)} f(p)\mathbf{1}_B(u)\widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,u)) = \int_B \frac{ds^{d-1}}{t^d} \int_{\mathscr{P}_k^o} f(p)\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)}(\mathrm{d}p)\mathrm{d}s. \tag{43}$$

For our purposes we need a slight generalization of these identities for arbitrary $j \in \{0, \ldots, k\}$. The proof of the generalized statement resembles the argument of

Lemma 4 in [**36**] for the length-weighted typical maximal segment $\mathsf{MP}_{1,1}^{(t)}$ of $Y(t)$ in $\mathbb{R}^3$ (in which $d=3$, $k=1$ and $j=1$).

Lemma 3.6.1. *Given $d \geq 2$, $k \in \{1,\dots,d-1\}$, $j \in \{0,\dots,k\}$, $t>0$ and $B_{d-k} \subset (0,t)$ a Borel set, it holds that*

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\big] = \int\limits_{B_{d-k}} \frac{(d-j)s^{d-j-1}}{t^{d-j}}\,\mathbb{E}f(\mathsf{F}_{k,j}^{(s)})\,\mathrm{d}s$$

for any non-negative measurable function $f : \mathscr{P}_k^o \to \mathbb{R}$.

Proof. For any Borel subset A of $\mathscr{P}_k^o$ and any Borel subset B_{d-k} of $(0,t)$, applying Proposition 1.4.12, Equation (43) and Proposition 3.4.2 in that order, we get

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_A(\mathsf{MP}_{k,j}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\big]\\
&= [\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1}\mathbb{E}[\mathbf{1}_A(\mathsf{MP}_{k,0}^{(t)})\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,0}^{(t)}))V_j(\mathsf{MP}_{k,0}^{(t)})]\\
&= [\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1}\int\limits_{\mathscr{P}_k^o\times(0,t)} \mathbf{1}_A(p)V_j(p)\mathbf{1}_{B_{d-k}}(\beta_{d-k})\widehat{\mathbb{P}}_{k,0}^{(t)}(\mathrm{d}(p,\beta_{d-k}))\\
&= [\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1}\int\limits_{B_{d-k}} \frac{ds^{d-1}}{t^d}\int\limits_{\mathscr{P}_k^o} \mathbf{1}_A(p)\,V_j(p)\,\mathbb{P}_{k,0}^{\mathrm{PHT}(s\Lambda)}(\mathrm{d}p)\,\mathrm{d}s\\
&= [\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1}\int\limits_{B_{d-k}} \frac{ds^{d-1}}{t^d}\mathbb{E}[\mathbf{1}_A(\mathsf{F}_{k,0}^{(s)})V_j(\mathsf{F}_{k,0}^{(s)})]\mathrm{d}s\\
&= [\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})]^{-1}\int\limits_{B_{d-k}} \frac{ds^{d-1}}{t^d}\mathbb{E}V_j(\mathsf{F}_{k,0}^{(s)})\mathbb{E}\mathbf{1}_A(\mathsf{F}_{k,j}^{(s)})\mathrm{d}s\\
&= \int\limits_{B_{d-k}} \frac{\mathbb{E}V_j(\mathsf{F}_{k,0}^{(s)})}{\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})}\frac{ds^{d-1}}{t^d}\mathbb{E}\mathbf{1}_A(\mathsf{F}_{k,j}^{(s)})\mathrm{d}s.
\end{aligned}$$

Denote by Π the associated zonoid of the stationary Poisson hyperplane process $\mathrm{PHP}(\Lambda)$. Using [**30**, Equation (10.3) and Theorem 10.3.3] together with [**31**, Corollary 4], we get

$$\mathbb{E}V_j(\mathsf{F}_{k,0}^{(s)}) = \frac{\binom{d-j}{d-k}V_{d-j}(s\Pi)}{\binom{d}{k}V_d(s\Pi)} = \frac{\binom{d-j}{d-k}s^{d-j}V_{d-j}(\Pi)}{\binom{d}{k}s^dV_d(\Pi)}$$

and

$$\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)}) = \frac{d}{d-j}\frac{\binom{d-j}{d-k}V_{d-j}(t\Pi)}{\binom{d}{k}V_d(t\Pi)} = \frac{d}{d-j}\frac{\binom{d-j}{d-k}t^{d-j}V_{d-j}(\Pi)}{\binom{d}{k}t^dV_d(\Pi)},$$

which implies

$$\frac{\mathbb{E}V_j(\mathsf{F}_{k,0}^{(s)})}{\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})} = \frac{\frac{\binom{d-j}{d-k}s^{d-j}V_{d-j}(\Pi)}{\binom{d}{k}s^dV_d(\Pi)}}{\frac{d}{d-j}\frac{\binom{d-j}{d-k}t^{d-j}V_{d-j}(\Pi)}{\binom{d}{k}t^dV_d(\Pi)}} = \frac{(d-j)t^j}{ds^j}.$$

We obtain, for any $A \in \mathcal{B}(\mathscr{P}_k^o)$ and $B_{d-k} \in \mathcal{B}((0,t))$,

$$\mathbb{E}\big[\mathbf{1}_A(\mathsf{MP}_{k,j}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\big] = \int\limits_{B_{d-k}} \frac{(d-j)s^{d-j-1}}{t^{d-j}}\,\mathbb{E}\mathbf{1}_A(\mathsf{F}_{k,j}^{(s)})\mathrm{d}s.$$

Thus the assertion holds for indicator functions of Borel sets of $\mathscr{P}_k^o$ and consequently also for linear combinations of such functions. By a standard argument of integration theory, it holds for any non-negative measurable function $f : \mathscr{P}_k^o \to \mathbb{R}$, that is,

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\big] = \int\limits_{B_{d-k}} \frac{(d-j)s^{d-j-1}}{t^{d-j}}\,\mathbb{E}f(\mathsf{F}_{k,j}^{(s)})\,\mathrm{d}s. \tag{44}$$

□

Remark 3.6.2. If $B_{d-k} = (0,t)$ then Equation (44) becomes

$$\mathbb{E}f(\mathsf{MP}_{k,j}^{(t)}) = \int\limits_0^t \frac{(d-j)s^{d-j-1}}{t^{d-j}}\,\mathbb{E}f(\mathsf{F}_{k,j}^{(s)})\,\mathrm{d}s. \tag{45}$$

for any non-negative measurable function $f : \mathscr{P}_k^o \to \mathbb{R}$. We observe that Equation (42) is a special case of Equation (45) when $j = 0$.

Example 3.6.3. To give a simple example of Equation (45), we consider the case $d = 3$, $k = 2$, $j = 0$. Now $\mathsf{MP}_{2,0}^{(t)}$ is the typical maximal polygon of the 3-dimensional STIT tessellation $Y(t)$. If $f : \mathscr{P}_2^o \to \mathbb{R}$ is non-negative and measurable, we get

$$\mathbb{E}f(\mathsf{MP}_{2,0}^{(t)}) = \int\limits_0^t \frac{3s^2}{t^3}\,\mathbb{E}f(\mathsf{F}_{2,0}^{(s)})\,\mathrm{d}s.$$

If $f(\cdot) = V_1(\cdot)$ then

$$\mathbb{E}V_1(\mathsf{MP}_{2,0}^{(t)}) = \int\limits_0^t \frac{3s^2}{t^3}\,\mathbb{E}V_1(\mathsf{F}_{2,0}^{(s)})\,\mathrm{d}s.$$

Using [**30**, Equation (10.3) and Theorem 10.3.3], we find that

$$\mathbb{E}V_1(\mathsf{F}_{2,0}^{(s)}) = \frac{\binom{3-1}{3-2}V_{3-1}(s\Pi)}{\binom{3}{2}V_3(s\Pi)} = \frac{2s^2V_2(\Pi)}{3s^3V_3(\Pi)} = \frac{2V_2(\Pi)}{3sV_3(\Pi)}.$$

Here Π is the associated zonoid of the stationary Poisson plane process with intensity measure Λ. We arrive at

$$\mathbb{E}V_1(\mathsf{MP}_{2,0}^{(t)}) = \int\limits_0^t \frac{3s^2}{t^3}\cdot\frac{2V_2(\Pi)}{3sV_3(\Pi)}\,\mathrm{d}s = \frac{V_2(\Pi)}{tV_3(\Pi)}.$$

In the isotropic case, i.e. when the probability measure $\mathbb{Q}$ on $G(3,2)$ in the decomposition (12) of Λ is rotation invariant, the zonoid Π is a ball. Thus $\Pi = rB^3$, where the radius r is determined by

$$r = \frac{\kappa_2}{3\kappa_3} = \frac{\pi}{3 \cdot \frac{4\pi}{3}} = \frac{1}{4},$$

recalling that $\kappa_i = \pi^{\frac{i}{2}}/\Gamma(1+\frac{i}{2})$. We obtain

$$\mathbb{E}V_1(\mathsf{MP}_{2,0}^{(t)}) = \frac{V_2(\frac{1}{4}B^3)}{tV_3(\frac{1}{4}B^3)} = \frac{\frac{1}{16}V_2(B^3)}{\frac{t}{64}V_3(B^3)} = \frac{\frac{1}{16}\binom{3}{2}\frac{\kappa_3}{\kappa_1}}{\frac{t}{64}\kappa_3} = \frac{6}{t}.$$

Consequently, the mean width of the typical maximal polygon of the 3-dimensional STIT tessellation $Y(t)$ is $3/t$.

Remark 3.6.4. The mean value $\mathbb{E}V_1(\mathsf{MP}_{2,0}^{(t)})$ in the case $d = 3$ is already obtained, for example, in [**31**] (with the same method).

Example 3.6.5. Our purpose is to calculate $\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)})$ for $n = 1, 2, \ldots$, namely, the nth moment of the length of the length-weighted typical maximal segment in the d-dimensional STIT tessellation $Y(t)$. Note that $\mathbb{E}V_1^{n+1}(\mathsf{MP}_{1,0}^{(t)})$ – the $(n+1)$th moment of the length of the typical maximal segment of $Y(t)$ – can be derived from $\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)})$ as follows (see Proposition 1.4.8):

$$\mathbb{E}V_1^{n+1}(\mathsf{MP}_{1,0}^{(t)}) = \mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)}) \cdot \mathbb{E}V_1(\mathsf{MP}_{1,0}^{(t)}).$$

Here, if Π denotes the associated zonoid of the stationary Poisson hyperplane process $\mathrm{PHP}(\Lambda)$ with intensity measure Λ then

$$\mathbb{E}V_1(\mathsf{MP}_{1,0}^{(t)}) = \frac{d}{d-1}\frac{\binom{d-1}{d-1}V_{d-1}(t\Pi)}{\binom{d}{1}V_d(t\Pi)} = \frac{dt^{d-1}V_{d-1}(\Pi)}{d(d-1)t^dV_d(\Pi)} = \frac{V_{d-1}(\Pi)}{(d-1)tV_d(\Pi)},$$

using [**31**, Corollary 4]. For $U \in \mathcal{B}(\mathscr{S}_+^{d-1})$, recall that $\mathcal{R}(U) = \mathbb{Q}(\{u^\perp : u \in U\})$, where $\mathbb{Q}$ is the probability measure in the decomposition (12) of Λ and $u^\perp$ denotes the orthogonal complement of the linear supspace spanned by u; see Example 1.1.5.

Proposition 3.6.6. *For $n = 1, 2, \ldots$, the nth moment of the length of the length-weighted typical maximal segment in the STIT tessellation $Y(t)$ is given by*

$$\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)}) = \frac{(n+1)!(d-1)}{t^{d-1}}\int_0^t s^{d-n-2}\mathrm{d}s \int_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^n}\,\mathbb{Q}_{d-1}(\mathrm{d}u),$$

Here if Π denotes the associated zonoid of the stationary Poisson hyperplane process $\mathrm{PHP}(\Lambda)$ and $[u_1, \ldots, u_{d-1}]$ denotes the $(d-1)$-dimensional volume of the parallelepiped spanned by $u_1, \ldots, u_{d-1}$ then

$$\mathbb{Q}_{d-1}(U) = \frac{1}{(d-1)!V_{d-1}(\Pi)}\int_{(\mathscr{S}_+^{d-1})^{d-1}} \mathbf{1}\{u_1^\perp \cap \ldots \cap u_{d-1}^\perp \cap \mathscr{S}_+^{d-1} \in U\}[u_1, \ldots, u_{d-1}] \,\mathcal{R}^{d-1}(\mathrm{d}(u_1, \ldots, u_{d-1})).$$

The mean value $\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)})$ is finite if and only if $n \le d-2$.

We need the following lemma to prove Proposition 3.6.6.

Lemma 3.6.7. *For $n = 1, 2, \ldots$, the nth moment of the length of the length-weighted typical edge in the Poisson hyperplane tessellation* $\mathrm{PHT}(s\Lambda)$ *is given by*

$$\mathbb{E}V_1^n(\mathsf{F}_{1,1}^{(s)}) = \frac{(n+1)!}{s^n} \int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^n}\, \mathbb{Q}_{d-1}(\mathrm{d}u).$$

Proof of Lemma 3.6.7. The intersection of the stationary Poisson hyperplane process $\mathrm{PHP}(s\Lambda)$ with a line $L_0 = \operatorname{span} u$ (where $u \in \mathscr{S}_+^{d-1}$) is a stationary Poisson process $\mathcal{O}$ in L_0 with intensity $\gamma_{\mathcal{O}}$ given by, using Theorem 1.1.7,

$$\gamma_{\mathcal{O}} = \mathbb{E}(\mathrm{PHP}(s\Lambda)([0,u])) = \mathbb{E} \sum_{\mathsf{H} \in \mathrm{PHP}(s\Lambda)} \mathbf{1}\{\mathsf{H} \cap [0,u] \neq \emptyset\}$$

$$= \int\limits_{A(d,d-1)} \mathbf{1}\{H \cap [0,u] \neq \emptyset\}(s\Lambda)(\mathrm{d}H) = (s\Lambda)(\{H : H \cap [0,u] \neq \emptyset\}) = \Lambda(\langle[0,u]\rangle)s.$$

Denote by $\mathcal{L}$ the (measurable) space of line segments in $\mathbb{R}^d$. Let $D : \mathcal{L} \to \mathscr{S}_+^{d-1}$ be a function that assigns to $L \in \mathcal{L}$ the unit vector $D(L) \in \mathscr{S}_+^{d-1}$ parallel to L. According to [**8**, Theorem 1], the conditional distribution of the length of the length-weighted typical edge $\mathsf{F}_{1,1}^{(s)}$ of $\mathrm{PHP}(s\Lambda)$, given that $D(\mathsf{F}_{1,1}^{(s)}) = u$, is equal to the distribution of the length of the interval containing the origin of the stationary Poisson process $\mathcal{O}$. The latter, according to [**9**, Equation (4.12)], is the Gamma distribution with parameter $(2, \Lambda(\langle[0,u]\rangle)s)$. Moreover, from [**8**, Theorem 1], we also see that, the distribution of $D(\mathsf{F}_{1,1}^{(s)})$ is $\mathbb{Q}_{d-1}$ (with the help of [**30**, Equation (4.63)]). Put $X := V_1(\mathsf{F}_{1,1}^{(s)})$. Let $\mathbb{P}_X$ be the distribution of X. For $x > 0$ we have

$$\mathbb{P}_X((0,x)) = \int\limits_{\mathscr{S}_+^{d-1}} \mathbb{P}_{X|D(\mathsf{F}_{1,1}^{(s)})=u}((0,x))\mathbb{Q}_{d-1}(\mathrm{d}u)$$

$$= \int\limits_{\mathscr{S}_+^{d-1}} \int\limits_0^x (\Lambda(\langle[0,u]\rangle)s)^2 y e^{-\Lambda(\langle[0,u]\rangle)sy}\, \mathrm{d}y\, \mathbb{Q}_{d-1}(\mathrm{d}u)$$

$$= \int\limits_{\mathscr{S}_+^{d-1}} \left[1 - \Lambda(\langle[0,u]\rangle)sxe^{-\Lambda(\langle[0,u]\rangle)sx} - e^{-\Lambda(\langle[0,u]\rangle)sx}\right] \mathbb{Q}_{d-1}(\mathrm{d}u).$$

Thus, $\mathbb{E}V_1^n(\mathsf{F}_{1,1}^{\mathrm{PHT}(s)}) = \mathbb{E}X^n = n \int\limits_0^{+\infty} x^{n-1}(1 - \mathbb{P}_X((0,x)))\, \mathrm{d}x$

$$= n \int\limits_0^{+\infty} x^{n-1}\left(1 - \int\limits_{\mathscr{S}_+^{d-1}} \left[1 - \Lambda(\langle[0,u]\rangle)sxe^{-\Lambda(\langle[0,u]\rangle)sx} - e^{-\Lambda(\langle[0,u]\rangle)sx}\right] \mathbb{Q}_{d-1}(\mathrm{d}u)\right)\mathrm{d}x$$

$$= n \int\limits_{\mathscr{S}_+^{d-1}} \int\limits_0^{+\infty} x^{n-1} \left[e^{-\Lambda(\langle [0,u] \rangle)sx} + \Lambda(\langle [0,u] \rangle) sx e^{-\Lambda(\langle [0,u] \rangle)sx} \right] \mathrm{d}x \, \mathbb{Q}_{d-1}(\mathrm{d}u)$$

$$= n \int\limits_{\mathscr{S}_+^{d-1}} \int\limits_0^{+\infty} x^{n-1} e^{-\Lambda(\langle [0,u] \rangle)sx} \mathrm{d}x \, \mathbb{Q}_{d-1}(\mathrm{d}u) +$$

$$+ n \int\limits_{\mathscr{S}_+^{d-1}} \int\limits_0^{+\infty} \Lambda(\langle [0,u] \rangle) s x^n e^{-\Lambda(\langle [0,u] \rangle)sx} \mathrm{d}x \, \mathbb{Q}_{d-1}(\mathrm{d}u)$$

By partial integration, we find that

$$\int\limits_0^{+\infty} \Lambda(\langle [0,u] \rangle) s x^n e^{-\Lambda(\langle [0,u] \rangle)sx} \mathrm{d}x = -x^n e^{-\Lambda(\langle [0,u] \rangle)sx} \Big|_0^{+\infty} + n \int\limits_0^{+\infty} x^{n-1} e^{-\Lambda(\langle [0,u] \rangle)sx} \, \mathrm{d}x$$

$$= n \int\limits_0^{+\infty} x^{n-1} e^{-\Lambda(\langle [0,u] \rangle)sx} \mathrm{d}x.$$

We obtain

$$\mathbb{E} V_1^n (\mathsf{F}_{1,1}^{\mathrm{PHT}(s)}) = (n + n^2) \int\limits_{\mathscr{S}_+^{d-1}} \int\limits_0^{+\infty} x^{n-1} e^{-\Lambda(\langle [0,u] \rangle)sx} \mathrm{d}x \, \mathbb{Q}_{d-1}(\mathrm{d}u)$$

$$= n(n+1) \int\limits_{\mathscr{S}_+^{d-1}} \frac{(n-1)!}{[\Lambda(\langle [0,u] \rangle)s]^n} \, \mathbb{Q}_{d-1}(\mathrm{d}u) = \frac{(n+1)!}{s^n} \int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle [0,u] \rangle)]^n} \, \mathbb{Q}_{d-1}(\mathrm{d}u).$$

□

Proof of Proposition 3.6.6. Combining Equation (45) and Lemma 3.6.7, we get

$$\mathbb{E} V_1^n (\mathsf{MP}_{1,1}^{(t)}) = \int\limits_0^t \frac{(d-1)s^{d-2}}{t^{d-1}} \, \mathbb{E} V_1^n (\mathsf{F}_{1,1}^{(s)}) \, \mathrm{d}s$$

$$= \frac{(n+1)!(d-1)}{t^{d-1}} \int\limits_0^t s^{d-n-2} \mathrm{d}s \int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle [0,u] \rangle)]^n} \, \mathbb{Q}_{d-1}(\mathrm{d}u).$$

□

If $d - n - 2 \leq -1$ or $d - n \leq 1$ then $\int\limits_0^t s^{d-n-2} \mathrm{d}s = +\infty$, which implies that $\mathbb{E} V_1^n (\mathsf{MP}_{1,1}^{(t)}) = +\infty$ and hence, $\mathbb{E} V_1^{n+1} (\mathsf{MP}_{1,0}^{(t)}) = +\infty$.

If $d-n>1$ then

$$\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)}) = \frac{(n+1)!(d-1)}{(d-n-1)t^{d-1}}\left(s^{d-n-1}\Big|_0^t\right)\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^n}\,\mathbb{Q}_{d-1}(\mathrm{d}u)$$

$$= \frac{(n+1)!(d-1)}{(d-n-1)t^{n}}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^n}\,\mathbb{Q}_{d-1}(\mathrm{d}u)$$

and consequently

$$\mathbb{E}V_1^{n+1}(\mathsf{MP}_{1,0}^{(t)}) = \frac{(n+1)!V_{d-1}(\Pi)}{t^{n+1}(d-n-1)V_d(\Pi)}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^n}\,\mathbb{Q}_{d-1}(\mathrm{d}u).$$

If $d\geq 3$ we have

$$\mathbb{E}V_1(\mathsf{MP}_{1,1}^{(t)}) = \frac{2(d-1)}{(d-2)t}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{\Lambda(\langle[0,u]\rangle)}\,\mathbb{Q}_{d-1}(\mathrm{d}u),$$

$$\mathbb{E}V_1^2(\mathsf{MP}_{1,0}^{(t)}) = \frac{2V_{d-1}(\Pi)}{(d-2)t^2V_d(\Pi)}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{\Lambda(\langle[0,u]\rangle)}\,\mathbb{Q}_{d-1}(\mathrm{d}u).$$

Hence, the variance of the length of the typical maximal segment $\mathsf{MP}_{1,0}^{(t)}$ of the STIT tessellation $Y(t)$, denoted by $\mathrm{Var}(V_1(\mathsf{MP}_{1,0}^{(t)}))$, is given by

$$\begin{aligned}\mathrm{Var}(V_1(\mathsf{MP}_{1,0}^{(t)})) &= \mathbb{E}V_1^2(\mathsf{MP}_{1,0}^{(t)}) - [\mathbb{E}V_1(\mathsf{MP}_{1,0}^{(t)})]^2\\ &= \frac{2V_{d-1}(\Pi)}{(d-2)t^2V_d(\Pi)}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{\Lambda(\langle[0,u]\rangle)}\,\mathbb{Q}_{d-1}(\mathrm{d}u) - \frac{V_{d-1}^2(\Pi)}{(d-1)^2t^2V_d^2(\Pi)}.\end{aligned}$$

Moreover, if $d\geq 4$ we get

$$\mathbb{E}V_1^2(\mathsf{MP}_{1,1}^{(t)}) = \frac{6(d-1)}{(d-3)t^2}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^2}\,\mathbb{Q}_{d-1}(\mathrm{d}u).$$

Therefore, we have

$$\mathrm{Var}(V_1(\mathsf{MP}_{1,1}^{(t)})) = \mathbb{E}V_1^2(\mathsf{MP}_{1,1}^{(t)}) - [\mathbb{E}V_1(\mathsf{MP}_{1,1}^{(t)})]^2$$

$$= \frac{6(d-1)}{(d-3)t^2}\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{[\Lambda(\langle[0,u]\rangle)]^2}\,\mathbb{Q}_{d-1}(\mathrm{d}u) - \frac{4(d-1)^2}{(d-2)^2t^2}\left(\int\limits_{\mathscr{S}_+^{d-1}} \frac{1}{\Lambda(\langle[0,u]\rangle)}\,\mathbb{Q}_{d-1}(\mathrm{d}u)\right)^2.$$

We consider the case that the probability measure $\mathbb{Q}$ on $G(d,d-1)$ in the decomposition (12) of Λ is rotation invariant. If ν_{d-1} denotes the unique rotation invariant probability measure on $G(d,d-1)$ (see [**30**, Theorem 13.2.11]), we infer that $\mathbb{Q}$ must be equal to ν_{d-1}. Therefore,

$$\begin{aligned}\Lambda(\langle[0,u]\rangle) &= \int\limits_{A(d,d-1)} \mathbf{1}\{H\cap[0,u]\neq\emptyset\}\Lambda(\mathrm{d}H)\\ &= \int\limits_{G(d,d-1)}\int\limits_{H_0^\perp} \mathbf{1}\{(H_0+x)\cap[0,u]\neq\emptyset\}\lambda_{H_0^\perp}(\mathrm{d}x)\nu_{d-1}(\mathrm{d}H_0)\\ &= \int\limits_{G(d,d-1)} \lambda_{H_0^\perp}([0,u]|H_0^\perp)\nu_{d-1}(\mathrm{d}H_0),\end{aligned}$$

where $[0,u]|H_0^\perp$ denotes the image of the interval $[0,u]$ under orthogonal projection to the subspace $H_0^\perp$. From the fact that the map $H_0\mapsto H_0^\perp$ transforms ν_{d-1} into ν_1, we get

$$\Lambda(\langle[0,u]\rangle) = \int\limits_{G(d,1)} \lambda_{H_0^\perp}([0,u]|H_0^\perp)\,\nu_1(\mathrm{d}H_0^\perp) = \frac{\kappa_1(d-1)!\kappa_{d-1}}{d!\kappa_d}V_1([0,u]) = \frac{2\kappa_{d-1}}{d\kappa_d}$$

using [**30**, Equation (5.8)]. Moreover, if $\mathbb{Q}$ is rotation invariant then $\Pi = rB^d$, where B_d is the unit ball of $\mathbb{R}^d$ and the radius r is determined by $r=\kappa_{d-1}/(d\kappa_d)$. In this case, if $d-n>1$ then

$$\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)}) = \frac{(n+1)!(d-1)d^n\kappa_d^n}{2^nt^n(d-n-1)\kappa_{d-1}^n} \text{ and } \mathbb{E}V_1^{n+1}(\mathsf{MP}_{1,0}^{(t)}) = \frac{(n+1)!d^{n+2}\kappa_d^{n+1}}{2^{n+1}t^{n+1}(d-n-1)\kappa_{d-1}^{n+1}}.$$

Furthermore, because $d-n-1>0$ implies that $d-n>0$, we get

$$\mathbb{E}V_1^n(\mathsf{MP}_{1,0}^{(t)}) = \frac{n!d^{n+1}\kappa_d^n}{2^nt^n(d-n)\kappa_{d-1}^n}.$$

Consequently, if $d-n>1$ and $n\geq 1$ then

$$\frac{\mathbb{E}V_1^n(\mathsf{MP}_{1,0}^{(t)})}{\mathbb{E}V_1^n(\mathsf{MP}_{1,1}^{(t)})} = \frac{1}{n+1} - \frac{n}{(d-n)(d-1)(n+1)} < \frac{1}{2}.$$

If $d\geq 3$ then

$$\mathrm{Var}(V_1(\mathsf{MP}_{1,0}^{(t)})) = \frac{d^3\kappa_d^2}{2t^2(d-2)\kappa_{d-1}^2} - \frac{d^4\kappa_d^2}{4t^2(d-1)^2\kappa_{d-1}^2} = \frac{d^3(d^2-2d+2)\kappa_d^2}{4(d-1)^2(d-2)t^2\kappa_{d-1}^2}.$$

In the case $d=2$ we have

$$\mathbb{E}V_1(\mathsf{MP}_{1,0}^{(t)}) = \frac{\pi}{t},\quad \mathbb{E}V_1(\mathsf{MP}_{1,1}^{(t)}) = \mathbb{E}V_1^2(\mathsf{MP}_{1,0}^{(t)}) = \mathrm{Var}(V_1(\mathsf{MP}_{1,0}^{(t)})) = +\infty.$$

If $d=3$ we obtain

$$\mathbb{E}V_1(\mathsf{MP}_{1,0}^{(t)}) = \frac{3}{t},\quad \mathbb{E}V_1(\mathsf{MP}_{1,1}^{(t)}) = \frac{8}{t},\quad \mathbb{E}V_1^2(\mathsf{MP}_{1,0}^{(t)}) = \frac{24}{t^2},\quad \mathrm{Var}(V_1(\mathsf{MP}_{1,0}^{(t)})) = \frac{15}{t^2}.$$

Remark 3.6.8. The second moment of the length of the typical maximal segment in the 3-dimensional STIT tessellation $Y(t)$, namely, $\mathbb{E}V_1^2(\mathsf{MP}_{1,0}^{(t)})$, is already obtained earlier in [**31**].

CHAPTER 4

Birth-time distributions of weighted polytopes in STIT tessellations

We recall that any k-dimensional maximal polytope p of the STIT tessellation $Y(t)$ is the intersection of $(d-k)$ maximal polytopes of dimension $(d-1)$. Each of these $(d-1)$-dimensional polytopes has a well-defined random birth-time. We denote the birth-times of these $(d-k)$ maximal polytopes by $\beta_1(\mathsf{p}), \ldots, \beta_{d-k}(\mathsf{p})$ and order them in such a way that $0 < \beta_1(\mathsf{p}) < \ldots < \beta_{d-k}(\mathsf{p}) < t$ holds almost surely. For $j \in \{0,, \ldots, k\}$, we compute the joint distribution of $(d-k)$ birth-times of the V_j-weighted typical k-dimensional maximal polytope of $Y(t)$ in this chapter.

4.1. Birth-time distributions of k-volume-weighted polytopes

Theorem 4.1.1. *Let $d \geq 2$ and $k \in \{0, \ldots, d-1\}$. The joint distribution of the birth-times $\beta_1(\mathsf{MP}_{k,k}^{(t)}), \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})$ of the k-volume-weighted typical k-maximal polytope $\mathsf{MP}_{k,k}^{(t)}$ of the STIT tessellation $Y(t)$ is the uniform distribution on $\Delta(t) = \{(r_1, \ldots, r_{d-k}) \in \mathbb{R}^{d-k} : 0 < r_1 < \ldots < r_{d-k} < t\}$, which has density*

$$p_{\beta_1(\mathsf{MP}_{k,k}^{(t)}), \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})}(s_1, \ldots, s_{d-k}) = \frac{(d-k)!}{t^{d-k}} \mathbf{1}\{0 < s_1 < \ldots < s_{d-k} < t\}.$$

To prepare for the proof of Theorem 4.1.1, we need the following lemma.

Lemma 4.1.2. *The k-volume density of the k-dimensional maximal polytopes of $Y(t)$ whose birth-times satisfy the constraints $\beta_1 \in (0, s_1), \ldots, \beta_{d-k} \in (s_{d-k-1}, s_{d-k})$, denoted by $\varrho_{k,k}^{(s_1, \ldots, s_{d-k}, t)}$, is given as follows*

$$\varrho_{k,k}^{(s_1, \ldots, s_{d-k}, t)} = \frac{1}{\lambda_d(B)} \times$$

$$\times \mathbb{E} \sum_{(c(\mathsf{p}), \mathsf{p}_o, \beta_1(\mathsf{p}_o), \ldots, \beta_{d-k}(\mathsf{p}_o)) \in \widetilde{\mathscr{M}\mathscr{P}}_k^{(t)}, c(\mathsf{p}) \in B} \mathbf{1}_{(0,s_1)}(\beta_1(\mathsf{p}_o)) \ldots \mathbf{1}_{(s_{d-k-1}, s_{d-k})}(\beta_{d-k}(\mathsf{p}_o)) V_k(\mathsf{p}_o)$$

$$= 2^{d-k-1} \prod_{j=1}^{d-k} (s_j - s_{j-1}) \int_{\mathscr{S}_+^{d-1}} \ldots \int_{\mathscr{S}_+^{d-1}} [u_1, \ldots, u_{d-k}] \mathcal{R}(\mathrm{d}u_1) \ldots \mathcal{R}(\mathrm{d}u_{d-k}),$$

where $s_0 := 0$, the notation $[u_1, \ldots, u_{d-k}]$ signifies the $(d-k)$-dimensional volume of the parallelepiped spanned by $u_1, \ldots, u_{d-k}$ and $\mathcal{R}(U)$ is already defined as $\mathbb{Q}(\{u^\perp : u \in U\})$ for $U \in \mathcal{B}(\mathscr{S}_+^{d-1})$. Recall that $\mathbb{Q}$ is the probability measure in the decomposition (12) of the hyperplane measure Λ of $Y(t)$.

To prepare for the proof of Lemma 4.1.2, we need some new notation and two further propositions. Fix $j \in \{1, 2, \ldots, d-k\}$. For each $(d-1)$-dimensional maximal polytope p_j of $Y(s_j - s_{j-1})$, we mark its circumcenter c_j with its direction u_j and the shifted polytope $\mathsf{p}_{jo} := \mathsf{p}_j - c(\mathsf{p}_j)$ whose circumcenter lies at the origin o. Note that $\mathsf{u}_j := \mathsf{H}_{jo}^{\perp} \cap \mathscr{S}_+^{d-1}$, where H_{jo} is the hyperplane containing p_{jo}. We obtain a marked point process, denoted by Φ_j, and call it the corresponding marked point process of $Y(s_j - s_{j-1})$. Instead of H_{jo}, we shall write $\mathsf{u}_j^{\perp}$. Put $\mathsf{p}(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo}) := \mathsf{p}_j$.

Proposition 4.1.3. *We have*

$$\int\limits_{\mathbb{R}^d} \ldots \int\limits_{\mathbb{R}^d} V_k(B \cap p(c_1, u_1, p_{1o}) \cap \ldots \cap p(c_{d-k}, u_{d-k}, p_{d-k,o}))\lambda_d(\mathrm{d}c_{d-k}) \ldots \lambda_d(\mathrm{d}c_1)$$

$$= [u_1, \ldots, u_{d-k}]\lambda_d(B)V_{d-1}(p_{1o}) \ldots V_{d-1}(p_{d-k,o}),$$

where $[u_1, \ldots, u_{d-k}]$ *is the* $(d-k)$*-dimensional volume of the parallelepiped spanned by* $u_1, \ldots, u_{d-k}$.

Proof. Indeed, fixing $j \in \{k, \ldots, d-2\}$ and $c_1, \ldots, c_{d-j-1} \in \mathbb{R}^d$ such that

$$\dim(B \cap p(c_1, u_1, p_{1o}) \cap \ldots \cap p(c_{d-j-1}, u_{d-j-1}, p_{d-j-1,o})) = j+1,$$

we get, using [**30**, Corollary 5.2.1],

$$\int\limits_{\mathbb{R}^d} V_j(B \cap p(c_1, u_1, p_{1o}) \cap \ldots \cap p(c_{d-j-1}, u_{d-j-1}, p_{d-j-1,o}) \cap p(c_{d-j}, u_{d-j}, p_{d-j,o}))\lambda_d(\mathrm{d}c_{d-j})$$

$$= \int\limits_{\mathbb{R}^d} V_j(B \cap p_1 \cap \ldots \cap p_{d-j-1} \cap (p_{d-j,o} + c_{d-j}))\lambda_d(\mathrm{d}c_{d-j})$$

$$= \sum_{i=j+1}^{d-1} V_j^{(i)}(B \cap p_1 \cap \ldots \cap p_{d-j-1}, p_{d-j,o});$$

see the definition of $V_j^{(i)}(B \cap p_1 \cap \ldots \cap p_{d-j-1}, p_{d-j,o})$ in [**30**, Corollary 5.2.1]. We notice that $V_j^{(i)}(B \cap p_1 \cap \ldots \cap p_{d-j-1}, p_{d-j,o}) = 0$ for all $i \geq j+2$. Furthermore

$$V_j^{(j+1)}(B \cap p_1 \cap \ldots \cap p_{d-j-1}, p_{d-j,o})$$

$$= [B \cap p_1 \cap \ldots \cap p_{d-j-1}, p_{d-j,o}]V_{j+1}(B \cap p_1 \cap \ldots \cap p_{d-j-1})V_{d-1}(p_{d-j,o})$$

$$= [p_1 \cap \ldots \cap p_{d-j-1}, p_{d-j,o}]V_{j+1}(B \cap p_1 \cap \ldots \cap p_{d-j-1})V_{d-1}(p_{d-j,o})$$

$$= [u_1^{\perp} \cap \ldots \cap u_{d-j-1}^{\perp}, u_{d-j}^{\perp}]V_{j+1}(B \cap p_1 \cap \ldots \cap p_{d-j-1})V_{d-1}(p_{d-j,o})$$

$$= \frac{[u_1^{\perp}, \ldots, u_{d-j}^{\perp}]}{[u_1^{\perp}, \ldots, u_{d-j-1}^{\perp}]}V_{j+1}(B \cap p_1 \cap \ldots \cap p_{d-j-1})V_{d-1}(p_{d-j,o})$$

$$= \frac{[u_1, \ldots, u_{d-j}]}{[u_1, \ldots, u_{d-j-1}]}V_{j+1}(B \cap p_1 \cap \ldots \cap p_{d-j-1})V_{d-1}(p_{d-j,o}).$$

Here $[p_1\cap\ldots\cap p_{d-j-1}, p_{d-j,o}] = [u_1^\perp\cap\ldots\cap u_{d-j-1}^\perp, u_{d-j}^\perp]$ due to the extensive definition of the subspace determinant; see [**30**, Page 183]. Moreover, since

$$\sum_{i=1}^{d-j} \dim u_j^\perp = (d-j)(d-1) \geq (d-j-1)d,$$

according to [**30**, Lemma 14.1.1] we have

$$[u_1^\perp \cap \ldots \cap u_{d-j-1}^\perp, u_{d-j}^\perp] = [u_1^\perp, \ldots, u_{d-j}^\perp]/[u_1^\perp, \ldots, u_{d-j-1}^\perp].$$

Finally, since

$$\sum_{i=1}^{d-j} \dim u_j^\perp \geq (d-j-1)d \quad \text{and} \quad \sum_{i=1}^{d-j-1} \dim u_j^\perp = (d-j-1)(d-1) > (d-j-2)d,$$

we obtain $[u_1^\perp, \ldots, u_{d-j}^\perp] = [u_1, \ldots, u_{d-j}]$ and $[u_1^\perp, \ldots, u_{d-j-1}^\perp] = [u_1, \ldots, u_{d-j-1}]$; see [**30**, Page 598]. Hence,

$$\int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} V_k(B \cap p(c_1, u_1, p_{1o}) \cap \ldots \cap p(c_{d-k}, u_{d-k}, p_{d-k,o}))\lambda_d(\mathrm{d}c_{d-k}) \ldots \lambda_d(\mathrm{d}c_1)$$

$$= \prod_{j=k}^{d-2} \frac{[u_1, \ldots, u_{d-j}]}{[u_1, \ldots, u_{d-j-1}]} V_{d-1}(p_{d-j,o}) \int_{\mathbb{R}^d} V_{d-1}(B\cap p(c_1, u_1, p_{1o}))\lambda_d(\mathrm{d}c_1)$$

$$= \frac{[u_1, \ldots, u_{d-k}]}{[u_1]} V_{d-1}(p_{d-k,o}) \ldots V_{d-1}(p_{2o}) \int_{\mathbb{R}^d} V_{d-1}(B\cap(p_{1o}+c_1))\lambda_d(\mathrm{d}c_1)$$

$$= [u_1, \ldots, u_{d-k}] V_{d-1}(p_{d-k,o}) \ldots V_{d-1}(p_{2o}) \lambda_d(B) V_{d-1}(p_{1o})$$

$$= [u_1, \ldots, u_{d-k}] \lambda_d(B) V_{d-1}(p_{1o}) \ldots V_{d-1}(p_{d-k,o}).$$

□

The result in Proposition 4.1.3 might also follow from [**37**, Theorem 5.1 and Equation (7.1)].

Proposition 4.1.4. *We have*

(i) *For $j = 1, \ldots, d-k$, the intensity of the marked point process Φ_j is*

$$\gamma_{\Phi_j} = \frac{s_j - s_{j-1}}{\mathbb{E}V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_j - s_{j-1})})}.$$

(ii) *The probability measure $\mathcal{R}$ satisfies*

$$\int_{\mathscr{S}_+^{d-1}} f(u_j)\mathcal{R}(\mathrm{d}u_j) = \frac{1}{\mathbb{E}V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_j - s_{j-1})})} \int_{\mathscr{S}_+^{d-1} \times \mathscr{P}_{d-1}^o} f(u_j) V_{d-1}(p_{jo}) \mathbb{Q}_j(\mathrm{d}(u_j, p_{jo}))$$

for any non-negative measurable function $f : \mathscr{S}_+^{d-1} \to \mathbb{R}$.

Proof. (i) The surface density of $Y(s_j - s_{j-1})$, namely the mean total $(d-1)$-volume of cell boundaries of $Y(s_j - s_{j-1})$ per unit d-volume, is equal to $s_j - s_{j-1}$, see [**36**, [**Interpretation of** t **and** $\mathcal{R}$]]. Therefore, for $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda_d(B) < \infty$,

$$s_j - s_{j-1} = \rho_{d-1,d-1}^{(s_j - s_{j-1})} = \frac{1}{\lambda_d(B)}\mathbb{E} \sum_{\mathsf{p} \in \mathscr{M}\mathscr{P}_{d-1}^{(s_j-s_{j-1})},\, c(\mathsf{p}) \in B} V_{d-1}(\mathsf{p}) = \gamma_{\Phi_j} \mathbb{E} V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_j - s_{j-1})}),$$

using Theorem 1.1.11(a) and the mean value identity $\gamma_{\mathscr{M}\mathscr{P}_{d-1}^{(s_j-s_{j-1})}} = \gamma_{\Phi_j}$.

(ii) We define a stationary random measure X; see [**30**, Section 3.1] for the definition of a random measure, as follows

$$\mathsf{X}(B) := \frac{1}{\lambda_d(B)} \sum_{(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo}) \in \Phi_j} \lambda_{\mathsf{p}(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo})}(B),$$

for $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda_d(B) < \infty$. Here $\lambda_{\mathsf{p}(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo})}(B) = \lambda_{d-1}(B \cap \mathsf{p}_j)$. Let $\mathbb{P}_{\mathsf{X}}^0$ be the Palm distribution of X. Furthermore, let $\overline{U}$ be the set of all realizations of the STIT tessellation $Y(s_j - s_{j-1})$ which satisfy the following property: the direction of the $(d-1)$-dimensional maximal polytope containing the origin o belongs to a Borel subset U of $\mathscr{S}_+^{d-1}$. According to [**21**, Corollary 2], $\mathcal{R}$ is the directional distribution of $Y(s_j - s_{j-1})$, i.e., the distribution of the normal direction in the typical boundary point of $Y(s_j - s_{j-1})$. We have, using [**30**, Theorem 3.3.2],

$$\int_{\mathscr{S}_+^{d-1}} \mathbf{1}_U(u_j) \mathcal{R}(\mathrm{d}u_j) = \mathcal{R}(U) = \mathbb{P}_{\mathsf{X}}^0(\overline{U}) = \frac{1}{\lambda_d(B)\gamma_{\mathsf{X}}} \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}_B(x) \mathbf{1}_{\overline{U}}(\mathsf{X} - x) \mathsf{X}(\mathrm{d}x)$$

$$= \frac{\mathbb{E} \sum_{(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo}) \in \Phi_j} \int_{\mathbb{R}^d} \mathbf{1}_B(x) \mathbf{1}_U(\mathsf{u}_j) \lambda_{\mathsf{p}(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo})}(\mathrm{d}x)}{\mathbb{E} \sum_{(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo}) \in \Phi_j} \int_{\mathbb{R}^d} \mathbf{1}_B(x) \lambda_{\mathsf{p}(\mathsf{c}_j, \mathsf{u}_j, \mathsf{p}_{jo})}(\mathrm{d}x)}$$

$$= \frac{\gamma_{\Phi_j} \int_{\mathscr{S}_+^{d-1} \times \mathscr{P}_{d-1}^o} \mathbf{1}_U(u_j) \int_{\mathbb{R}^d} V_{d-1}(B \cap \mathsf{p}(c_j, u_j, p_{jo})) \lambda_d(\mathrm{d}c_j) \mathbb{Q}_j(\mathrm{d}(u_j, p_{jo}))}{\gamma_{\Phi_j} \int_{\mathscr{S}_+^{d-1} \times \mathscr{P}_{d-1}^o} \int_{\mathbb{R}^d} V_{d-1}(B \cap \mathsf{p}(c_j, u_j, p_{jo})) \lambda_d(\mathrm{d}c_j) \mathbb{Q}_j(\mathrm{d}(u_j, p_{jo}))}$$

$$= \frac{\lambda_d(B) \int_{\mathscr{S}_+^{d-1} \times \mathscr{P}_{d-1}^o} \mathbf{1}_U(u_j) V_{d-1}(p_{jo}) \mathbb{Q}_j(\mathrm{d}(u_j, p_{jo}))}{\lambda_d(B) \int_{\mathscr{S}_+^{d-1} \times \mathscr{P}_{d-1}^o} V_{d-1}(p_{jo}) \mathbb{Q}_j(\mathrm{d}(u_j, p_{jo}))}$$

$$= \frac{1}{\mathbb{E} V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_j - s_{j-1})})} \int_{\mathscr{S}_+^{d-1} \times \mathscr{P}_{d-1}^o} \mathbf{1}_U(u_j) V_{d-1}(p_{jo}) \mathbb{Q}_j(\mathrm{d}(u_j, p_{jo})).$$

We have proved that the assertion holds for indicator functions of Borel sets and hence also for linear combinations of indicator functions. By a standard argument of integration theory, it holds for all non-negative measurable functions $f : \mathscr{S}_+^{d-1} \to \mathbb{R}$. $\square$

Proof of Lemma 4.1.2. By Equation (18), we know that

$$Y(t) \overset{d}{=} Y(s_1) \boxplus Y(s_2 - s_1) \boxplus \ldots \boxplus Y(s_{d-k} - s_{d-k-1}) \boxplus Y(t - s_{d-k}),$$

For $j = 1, 2, \ldots, d-k-1$, write

$$Y(s_1) \boxplus Y(s_2 - s_1) \boxplus \ldots \boxplus Y(s_j - s_{j-1}) = \{\mathsf{z}_{ji_j} : i_j = 1, 2, \ldots\}.$$

Consider a cell z_{ji_j} of $Y(s_1) \boxplus Y(s_2 - s_1) \boxplus \ldots \boxplus Y(s_j - s_{j-1})$. Let Φ_{j+1,i_j} be the corresponding marked point process of a copy $Y_{i_j}(s_{j+1} - s_j)$ of $Y(s_{j+1} - s_j)$ satisfying that $Y_{i_j}(s_{j+1} - s_j)$ is locally superimposed within the cell z_{ji_j}. Note that the marked point process Φ_{j+1,i_j} of $Y_{i_j}(s_{j+1} - s_j)$ are obtained in a similar way as Φ_{j+1}, $j = 1, 2, \ldots, d-k-1$. By definition, for fixed $j \in \{1, 2, \ldots, d-k-1\}$, the copies $Y_{i_j}(s_{j+1} - s_j)$, $i_j = 1, 2, \ldots$ are i.i.d therefore Φ_{j+1,i_j}, $i_j = 1, 2, \ldots$ are independent and have the same distribution as the distribution of the corresponding marked point process Φ_{j+1} of $Y(s_{j+1} - s_j)$.

Recall that any k-dimensional maximal polytope of $Y(t)$ is the intersection of $(d-k)$ maximal polytopes of dimension $(d-1)$. We consider the k-maximal polytopes whose the first corresponding $(d-1)$-maximal polytope p_1 is born during the time interval $(0, s_1)$, the second corresponding $(d-1)$-maximal polytope p_2 is born during the time interval (s_1, s_2) ... the $(d-k)$th or the last corresponding $(d-1)$-maximal polytope p_{d-k} is born during the time interval (s_{d-k-1}, s_{d-k}).

At the first step, the hyperplane containing p_1 divides $\mathbb{R}^d$ into 2 parts which are denoted by p_1^+ and p_1^-. At the second step, p_2 could appear in p_1^+ or p_1^-. Without loss of generality, assume that p_2 appears in p_1^+. The restriction of the hyperplane containing p_2 to p_1^+ divides p_1^+ into 2 parts denoted by p_2^+ and p_2^-. Therefore, $\mathbb{R}^d$ is divided into 3 parts, namely, p_1^-, p_2^- and p_2^+. Because by definition, a k-dimensional maximal polytope must be a k-dimensional face of a $(d-1)$-dimensional maximal polytope for any $k = 0, \ldots, d-2$, we emphasize that at the third step, p_3 could only appear in p_2^+ or p_2^-. Without loss of generality, assume that p_3 appears in p_2^+. The restriction of the hyperplane containing p_3 to p_2^+ divides p_2^+ into 2 parts denoted by p_3^+ and p_3^-. Hence, $\mathbb{R}^d$ is divided into 4 parts which are p_1^-, p_2^-, p_3^- and p_3^+. At the fourth step, p_4 could only appear in p_3^+ or p_3^-. In gereral, at the $(j+1)$th step, p_{j+1}, $j = 1, 2, \ldots, d-k-2$, could only appear in p_j^+ or p_j^-. Without loss of generality, assume that p_{j+1} appears in p_j^+. The restriction of the hyperplane containing p_{j+1} to p_j^+ divides p_j^+ into 2 parts denoted by p_{j+1}^+ and p_{j+1}^-. As a consequence, $\mathbb{R}^d$ is divided into $(j+2)$ parts which are $\mathsf{p}_1^-, \ldots, \mathsf{p}_j^-, \mathsf{p}_{j+1}^-, \mathsf{p}_{j+1}^+$. At the $(d-k)$th step, p_{d-k} could only appear in p_{d-k-1}^+ or p_{d-k-1}^-. Further for $j = 1, \ldots, d-k-1$, at the jth step, we decompose the $(d-1)$-dimensional maximal polytope p_j into $(d-1)$-dimensional parts such that each part of p_j is the intersection of two d-dimensional cells of the STIT tessellation $Y(s_1) \boxplus Y(s_2 - s_1) \boxplus \ldots \boxplus Y(s_j - s_{j-1})$. For such a part of p_j which is denoted by p_j^*, one d-dimensional cell lies in p_j^+, denoted by z_{jm} and the other lies in p_j^-, denoted by z_{jn} with $m, n \in \mathbb{N}$ and $m \neq n$. Since $Y_{i_j}(s_{j+1} - s_j)$, $i_j = 1, 2, \ldots$ are i.i.d then $Y_m(s_{j+1} - s_j)$ and $Y_n(s_{j+1} - s_j)$ have the same distribution. We derive that $Y_m(s_{j+1} - s_j) \cap \mathsf{p}_j^*$ and $Y_n(s_{j+1} - s_j) \cap \mathsf{p}_j^*$ have the same distribution. We repeat the argument for all parts of p_j. Consequently, since the functional V_k is additive (see [**30**, Page 600]), if we work in a fixed part of p_j which will be assumed to be p_j^+, the jth step contributes a factor 2 to $\varrho_{k,k}^{(s_1,\ldots,s_{d-k},t)}$ for $j=1,\ldots,d-k-1$. Therefore, we have the factor 2^{d-k-1} in the representation of $\varrho_{k,k}^{(s_1,\ldots,s_{d-k},t)}$ as we see in the next computation.

The realization φ_{1i_0} of Φ_1 determines the corresponding realization $y(s_1)$ of $Y(s_1)$ uniquely. Similarly, the realization φ_{j+1,i_j} of Φ_{j+1,i_j} determines the corresponding realization $y_{i_j}(s_{j+1}-s_j)$ of $Y_{i_j}(s_{j+1}-s_j)$ uniquely for $j=1,2\ldots,d-k-2$. With the help of [**30**, Theorem 4.5.1], this leads to

$$\begin{aligned}
&\varrho_{k,k}^{(s_1,\ldots,s_{d-k},t)}\\
&= \frac{2^{d-k-1}}{\lambda_d(B)}\int \sum_{(c_1,u_1,p_{1o})\in\varphi_{1i_0}} \; \sum_{z_{1i_1}\in M_1\cap p_1^+} \int \sum_{(c_2,u_2,p_{2o})\in\varphi_{2i_1}} \; \sum_{z_{2i_2}\in M_2\cap p_2^+} \int \sum_{(c_3,u_3,p_{3o})\in\varphi_{3i_2}}\\
&\quad \sum_{\mathsf{z}_{3i_3}\in M_3\cap p_3^+} \cdots \int \sum_{(c_{d-k-1},u_{d-k-1},p_{d-k-1,o})\in\varphi_{d-k-1,i_{d-k-2}}} \; \sum_{z_{d-k-1,i_{d-k-1}}\in M_{d-k-1}\cap p_{d-k-1}^+}\\
&\quad \int \sum_{(c_{d-k},u_{d-k},p_{d-k,o})\in\varphi_{d-k,i_{d-k-1}}} V_k(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_{d-k}\cap\partial z_{d-k-1,i_{d-k-1}})\\
&\quad \mathbb{P}_{\Phi_{d-k}}(\mathrm{d}\varphi_{d-k,i_{d-k-1}})\mathbb{P}_{\Phi_{d-k-1}}(\mathrm{d}\varphi_{d-k-1,i_{d-k-2}})\ldots\mathbb{P}_{\Phi_3}(\mathrm{d}\varphi_{3i_2})\mathbb{P}_{\Phi_2}(\mathrm{d}\varphi_{2i_1})\mathbb{P}_{\Phi_1}(\mathrm{d}\varphi_{1i_0})
\end{aligned}$$

where $B\in\mathcal{B}(\mathbb{R}^d)$ with $0<\lambda_d(B)<\infty$ and ∂p is the boundary of a polytope p in $\mathbb{R}^d$. Here we define $M_1:=\{z_{1i_1}\in y(s_1):\dim(z_{1i_1}\cap p_1)=d-1\}$ and

$$\begin{aligned}
M_{j+1}:=\{z_{j+1,i_{j+1}}\in z_{ji_j}\cap y_{i_j}(s_{j+1}-s_j):\dim(&z_{j+1,i_{j+1}}\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_{j+1}\cap\partial z_{ji_j})\\
&=d-j-1\}
\end{aligned}$$

for $j=1,\ldots,d-k-2$, where

$$z_{ji_j}\cap y_{i_j}(s_{j+1}-s_j):=\{z_{ji_j}\cap z: z\in y_{i_j}(s_{j+1}-s_j),\mathrm{int}\, z_{ji_j}\cap\mathrm{int}\, z\neq\emptyset\}.$$

On the other hand, using Theorem 1.1.15,

$$\begin{aligned}
&\int \sum_{(c_{d-k},u_{d-k},p_{d-k,o})\in\varphi_{d-k,i_{d-k-1}}} V_k(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_{d-k}\cap\partial z_{d-k-1,i_{d-k-1}})\\
&\hspace{20em}\mathbb{P}_{\Phi_{d-k}}(\mathrm{d}\varphi_{d-k,i_{d-k-1}})\\
&= \int_{\mathbb{R}^d\times\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o} V_k(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p(c_{d-k},u_{d-k},p_{d-k,o})\cap\partial z_{d-k-1,i_{d-k-1}})\\
&\hspace{20em}\Theta_{d-k}(\mathrm{d}(c_{d-k},u_{d-k},p_{d-k,o})),
\end{aligned}$$

Moreover, put $p_1\cap\partial z_{0i_0}:=p_1$. For $j=d-k-1,\ldots,2,1$ and $p_{j+1},p_{j+2},\ldots,p_{d-k}$ being polytopes of $\mathbb{R}^d$, we have

$$\begin{aligned}
&\int \sum_{(c_j,u_j,p_{jo})\in\varphi_{ji_{j-1}}} \; \sum_{z_{ji_j}\in M_j\cap p_j^+} V_k(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_j\cap\\
&\qquad\cap\partial z_{j-1,i_{j-1}}\cap p_{j+1}\cap\partial z_{ji_j}\cap p_{j+2}\cap p_{j+3}\cap\ldots\cap p_{d-k})\mathbb{P}_{\Phi_j}(\mathrm{d}\varphi_{j,i_{j-1}})
\end{aligned}$$

$$= \int \sum_{(c_j,u_j,p_{jo})\in\varphi_{ji_{j-1}}} V_k\Bigg(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_j\cap\partial z_{j-1,i_{j-1}}\cap$$

$$\cap\Big(\bigcup_{z_{ji_j}\in M_j\cap p_j^+}\partial z_{ji_j}\Big)\cap p_{j+1}\cap p_{j+2}\cap\ldots\cap p_{d-k}\Bigg)\mathbb{P}_{\Phi_j}(\mathrm{d}\varphi_{ji_{j-1}})$$

$$= \int \sum_{(c_j,u_j,p_{jo})\in\varphi_{ji_{j-1}}} V_k(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_j\cap\partial z_{j-1,i_{j-1}}\cap p_{j+1}\cap p_{j+2}\cap\ldots\cap p_{d-k})$$

$$\mathbb{P}_{\Phi_j}(\mathrm{d}\varphi_{ji_{j-1}})$$

$$= \int\limits_{\mathbb{R}^d\times\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o} V_k(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p(c_j,u_j,p_{jo})\cap\partial z_{j-1,i_{j-1}}\cap p_{j+1}\cap p_{j+2}\cap\ldots\cap p_{d-k})$$

$$\varTheta_j(\mathrm{d}(c_j,u_j,p_{jo})).$$

Here we have used the following two facts. The first one is

$$\dim(B\cap p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_j\cap\partial z_{j-1m}\cap\partial z_{j-1n}\cap p_{j+1}\cap p_{j+2}\cap\ldots\cap p_{d-k})<k$$

for $z_{j-1m}, z_{j-1n}\in M_j\cap p_j^+$, $m\neq n$. The second one is

$$p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_j\cap\partial z_{j-1,i_{j-1}}\cap\Big(\bigcup_{z_{ji_j}\in M_j\cap p_j^+}\partial z_{ji_j}\Big)=p_1\cap p_2\cap\partial z_{1i_1}\cap\ldots\cap p_j\cap\partial z_{j-1i_{j-1}}.$$

Consequently, using Theorem 1.1.15,

$$\varrho_{k,k}^{(s_1,\ldots,s_{d-k},t)} =$$

$$\frac{2^{d-k-1}}{\lambda_d(B)}\int\limits_{\mathbb{R}^d\times\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\cdots\int\limits_{\mathbb{R}^d\times\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o} V_k(B\cap p(c_1,u_1,p_{1o})\cap\ldots\cap p(c_{d-k},u_{d-k},p_{d-k,o}))$$

$$\varTheta_1(\mathrm{d}(c_1,u_1,p_{1o}))\ldots\varTheta_{d-k}(\mathrm{d}(c_{d-k},u_{d-k},p_{d-k,o}))$$

$$=\frac{2^{d-k-1}}{\lambda_d(B)}\gamma_{\Phi_1}\ldots\gamma_{\Phi_{d-k}}\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\cdots\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} V_k(B\cap p(c_1,u_1,p_{1o})\cap\ldots$$

$$\cap p(c_{d-k},u_{d-k},p_{d-k,o}))\lambda_d(\mathrm{d}c_{d-k})\ldots\lambda_d(\mathrm{d}c_1)\mathbb{Q}_1(\mathrm{d}(u_1,p_{1o}))\ldots\mathbb{Q}_{d-k}(\mathrm{d}(u_{d-k},p_{d-k,o})),$$

where $\gamma_{\Phi_1},\ldots,\gamma_{\Phi_{d-k}}$ and $\mathbb{Q}_1,\ldots,\mathbb{Q}_{d-k}$ are the intensities and the mark distributions of $\Phi_1,\ldots,\Phi_{d-k}$, respectively.

From Proposition 4.1.3 we infer that the k-volume density of the k-dimensional maximal polytopes of $Y(t)$ whose birth-times satisfy the constraints $\beta_1\in(0,s_1),\ldots,$ $\beta_{d-k}\in(s_{d-k-1},s_{d-k})$, namely $\varrho_{k,k}^{(s_1,\ldots,s_{d-k},t)}$, is

$$2^{d-k-1}\gamma_{\Phi_1}\ldots\gamma_{\Phi_{d-k}}\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\cdots\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}[u_1,\ldots,u_{d-k}]V_{d-1}(p_{1o})\ldots V_{d-1}(p_{d-k,o})$$

$$\mathbb{Q}_1(\mathrm{d}(u_1,p_{1o}))\ldots\mathbb{Q}_{d-k}(\mathrm{d}(u_{d-k},p_{d-k,o})).$$

Using Proposition 4.1.4(i), we get

$$\varrho_{k,k}^{(s_1,\dots,s_{d-k},t)} = 2^{d-k-1}\prod_{j=1}^{d-k}\frac{s_j - s_{j-1}}{\mathbb{E}V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_j-s_{j-1})})}\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\dots\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}[u_1,\dots,u_{d-k}]\times$$

$$\times V_{d-1}(p_{1o})\dots V_{d-1}(p_{d-k,o})\mathbb{Q}_1(\mathrm{d}(u_1,p_{1o}))\dots\mathbb{Q}_{d-k}(\mathrm{d}(u_{d-k},p_{d-k,o})).$$

We observe that, for $j = 0,1,\dots,d-k-1$,

$$\frac{1}{\mathbb{E}V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_{j+1}-s_j)})}\underbrace{\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\dots\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}}_{(d-k-j)\text{ times}}\ \underbrace{\int\limits_{\mathscr{S}_+^{d-1}}\dots\int\limits_{\mathscr{S}_+^{d-1}}}_{j\text{ times}}[u_1,\dots,u_{d-k}]\mathcal{R}(\mathrm{d}u_1)$$

$$\dots\mathcal{R}(\mathrm{d}u_j)V_{d-1}(p_{j+1,o})\dots V_{d-1}(p_{d-k,o})\mathbb{Q}_{j+1}(\mathrm{d}(u_{j+1},p_{j+1,o}))\dots\mathbb{Q}_{d-k}(\mathrm{d}(u_{d-k},p_{d-k,o}))$$

$$=\underbrace{\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\dots\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}}_{(d-k-j-1)\text{ times}}\frac{1}{\mathbb{E}V_{d-1}(\mathsf{MP}_{d-1,0}^{(s_{j+1}-s_j)})}\times$$

$$\times\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\underbrace{\int\limits_{\mathscr{S}_+^{d-1}}\dots\int\limits_{\mathscr{S}_+^{d-1}}}_{j\text{ times}}[u_1,\dots,u_{d-k}]\mathcal{R}(\mathrm{d}u_1)\dots\mathcal{R}(\mathrm{d}u_j)V_{d-1}(p_{j+1,o})\mathbb{Q}_{j+1}(\mathrm{d}(u_{j+1},p_{j+1,o}))$$

$$V_{d-1}(p_{j+2,o})\dots V_{d-1}(p_{d-k,o})\mathbb{Q}_{j+2}(\mathrm{d}(u_{j+2},p_{j+2,o}))\dots\mathbb{Q}_{d-k}(\mathrm{d}(u_{d-k},p_{d-k,o}))$$

$$=\underbrace{\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}\dots\int\limits_{\mathscr{S}_+^{d-1}\times\mathscr{P}_{d-1}^o}}_{(d-k-j-1)\text{ times}}\ \underbrace{\int\limits_{\mathscr{S}_+^{d-1}}\dots\int\limits_{\mathscr{S}_+^{d-1}}}_{(j+1)\text{ times}}[u_1,\dots,u_{d-k}]\mathcal{R}(\mathrm{d}u_1)\dots\mathcal{R}(\mathrm{d}u_{j+1})$$

$$V_{d-1}(p_{j+2,o})\dots V_{d-1}(p_{d-k,o})\mathbb{Q}_{j+2}(\mathrm{d}(u_{j+2},p_{j+2,o}))\dots\mathbb{Q}_{d-k}(\mathrm{d}(u_{d-k},p_{d-k,o})).$$

We have used Proposition 4.1.4(ii) for the last equality. We obtain

$$\varrho_{k,k}^{(s_1,\dots,s_{d-k},t)} = 2^{d-k-1}\prod_{j=1}^{d-k}(s_j - s_{j-1})\int\limits_{\mathscr{S}_+^{d-1}}\dots\int\limits_{\mathscr{S}_+^{d-1}}[u_1,\dots,u_{d-k}]\mathcal{R}(\mathrm{d}u_1)\dots\mathcal{R}(\mathrm{d}u_{d-k}),$$

recalling our convention that $s_0 = 0$. $\square$

Proof of Theorem 4.1.1. In order to determine the joint distribution of the birth-times $\beta_1(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})$ we are now going to calculate the probability

$$\mathbb{P}\big(\beta_1(\mathsf{MP}_{k,k}^{(t)})\in(0,s_1),\dots,\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})\in(s_{d-k-1},s_{d-k})\big), \tag{46}$$

where $0 < s_1 < \dots < s_{d-k} < t$ are fixed. By definition, for $B\in\mathcal{B}(\mathbb{R}^d)$ with $0<\lambda_d(B)<\infty$,

$$\mathbb{P}\big(\beta_1(\mathsf{MP}_{k,k}^{(t)})\in(0,s_1),\dots,\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})\in(s_{d-k-1},s_{d-k})\big)$$

$$=\mathbb{P}_{\mathsf{MP}_{k,k}^{(t)},\beta_1(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})}(\mathscr{P}_k^o\times(0,s_1)\times\dots\times(s_{d-k-1},s_{d-k}))=$$

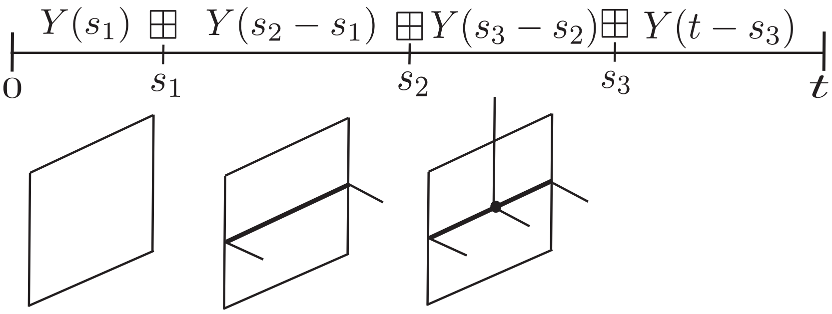

FIGURE 11. Illustration for the proof of Theorem 4.1.1. Here $d = 3$ and $k = 0$.

$$= \frac{\mathbb{E} \sum_{(c(\mathsf{p}),\mathsf{p}_o,\beta_1(\mathsf{p}_o),\ldots,\beta_{d-k}(\mathsf{p}_o))\in\widetilde{\mathscr{M}\mathscr{P}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\mathbf{1}_{(0,s_1)}(\beta_1(\mathsf{p}_o))\ldots\mathbf{1}_{(s_{d-k-1},s_{d-k})}(\beta_{d-k}(\mathsf{p}_o))V_k(\mathsf{p}_o)}{\mathbb{E}\sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}}\mathbf{1}_B(c(\mathsf{p}))V_k(\mathsf{p})}$$

We have

$$\frac{1}{\lambda_d(B)}\mathbb{E}\sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}}\mathbf{1}_B(c(\mathsf{p}))V_k(\mathsf{p}) = \varrho_{k,k}^{(t)} = t^{d-k}\varrho_{k,k}^{(1)}$$

according to Lemma 1.4.17a). Hence

$$\mathbb{P}\big(\beta_1(\mathsf{MP}_{k,k}^{(t)}) \in (0,s_1),\ldots,\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) \in (s_{d-k-1},s_{d-k})\big) = \frac{\varrho_{k,k}^{(s_1,\ldots,s_{d-k},t)}}{\varrho_{k,k}^{(t)}} =$$

$$= \frac{2^{d-k-1}\int_{\mathscr{S}_+^{d-1}}\cdots\int_{\mathscr{S}_+^{d-1}}[u_1,\ldots,u_{d-k}]\mathcal{R}(\mathrm{d}u_1)\ldots\mathcal{R}(\mathrm{d}u_{d-k})}{\rho_{k,k}^{(1)}} \times \frac{\prod_{j=1}^{d-k}(s_j - s_{j-1})}{t^{d-k}}.$$

Write $(\beta_1,\ldots,\beta_{d-k}) := (\beta_1(\mathsf{MP}_{k,k}^{(t)}),\ldots,\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}))$. To derive the formula for the joint density, we observe that – with $0 < s_1 < \ldots < s_{d-k} < t$ – the cumulative distribution function of the random variables $\beta_1,\ldots,\beta_{d-k}$ may be expressed as

$$\mathbb{P}\big(\beta_1 \in (0,s_1),\beta_2\in(0,s_2),\ldots,\beta_{d-k}\in(0,s_{d-k})\big)$$

$$= \sum_{j=1}^{d-k-1}\mathbb{P}\big[\beta_1\in(0,s_1),\beta_2\in(s_1,s_2),\ldots,\beta_j\in(s_{j-1},s_j),\beta_{j+1}\in(0,s_j),\beta_{j+2}\in(0,s_{j+2}),$$
$$\beta_{j+3}\in(0,s_{j+3}),\ldots,\beta_{d-k}\in(0,s_{d-k})\big]+$$
$$+\mathbb{P}\big(\beta_1\in(0,s_1),\beta_2\in(s_1,s_2),\ldots,\beta_{d-k}\in(s_{d-k-1},s_{d-k})\big).$$

For example, if $d = 3$ and $k = 0$ and $0 < s_1 < s_2 < s_3 < t$ then $(\beta_1,\beta_2,\beta_3) := (\beta_1(\mathsf{MP}_{0,0}^{(t)}),\beta_2(\mathsf{MP}_{0,0}^{(t)}),\beta_3(\mathsf{MP}_{0,0}^{(t)}))$. It is easy to see that

$$\mathbb{P}\big(\beta_1\in(0,s_1),\beta_2\in(0,s_2),\beta_3\in(0,s_3)\big) = \mathbb{P}\big(\beta_1\in(0,s_1),\beta_2\in(0,s_1),\beta_3\in(0,s_3)\big)+$$
$$+\mathbb{P}\big(\beta_1\in(0,s_1),\beta_2\in(s_1,s_2),\beta_3\in(0,s_2)\big)+\mathbb{P}\big(\beta_1\in(0,s_1),\beta_2\in(s_1,s_2),\beta_3\in(s_2,s_3)\big).$$

We notice that for $j = 1, 2, \dots, d-k-1$, the value $\mathbb{P}\big[\beta_1 \in (0, s_1), \beta_2 \in (s_1, s_2), \dots, \beta_j \in (s_{j-1}, s_j), \beta_{j+1} \in (0, s_j), \beta_{j+2} \in (0, s_{j+2}), \beta_{j+3} \in (0, s_{j+3}), \dots, \beta_{d-k} \in (0, s_{d-k})\big]$ does not depend on s_{j+1}.

Thus, only the derivative of $\mathbb{P}\big(\beta_1 \in (0, s_1), \beta_2 \in (s_1, s_2), \dots, \beta_{d-k} \in (s_{d-k-1}, s_{d-k})\big)$ remains after differentiation of $\mathbb{P}\big(\beta_1 \in (0, s_1), \beta_2 \in (0, s_2), \dots, \beta_{d-k} \in (0, s_{d-k})\big)$ with respect to $s_1, \dots, s_{d-k}$, and it equals

$$\frac{2^{d-k-1} \int\limits_{\mathscr{S}_+^{d-1}} \dots \int\limits_{\mathscr{S}_+^{d-1}} [u_1, \dots, u_{d-k}] \mathcal{R}(\mathrm{d}u_1) \dots \mathcal{R}(\mathrm{d}u_{d-k})}{\rho_{k,k}^{(1)}} t^{-(d-k)}.$$

Since this integrates to 1, we must have

$$\frac{2^{d-k-1} \int\limits_{\mathscr{S}_+^{d-1}} \dots \int\limits_{\mathscr{S}_+^{d-1}} [u_1, \dots, u_{d-k}] \mathcal{R}(\mathrm{d}u_1) \dots \mathcal{R}(\mathrm{d}u_{d-k})}{\rho_{k,k}^{(1)}} = (d-k)! \tag{47}$$

and hence, the joint distribution of the birth-times $\beta_1(\mathsf{MP}_{k,k}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})$ of the k-volume-weighted typical k-dimensional maximal polytope $\mathsf{MP}_{k,k}^{(t)}$ of $Y(t)$ is the uniform distribution on the simplex $\Delta(t)$, which has density

$$(s_1, \dots, s_{d-k}) \mapsto \frac{(d-k)!}{t^{d-k}} \mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\}.$$

$\square$

Remark 4.1.5. It is worth pointing out that the density

$$s_{d-k} \mapsto (d-k) s_{d-k}^{d-k-1} t^{-(d-k)} \mathbf{1}\{0 < s_{d-k} < t\}$$

is the marginal density of the last birth-time $\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})$ of the k-volume-weighted typical maximal polytope of dimension k in Theorem 4.1.1.

Corollary 4.1.6. *The k-volume density of $\mathscr{MP}_k^{(1)}$ in the STIT tessellation $Y(1)$, namely, $\rho_{k,k}^{(1)}$, is given by*

$$\rho_{k,k}^{(1)} = \frac{2^{d-k-1}}{(d-k)!} \int\limits_{\mathscr{S}_+^{d-1}} \dots \int\limits_{\mathscr{S}_+^{d-1}} [u_1, \dots, u_{d-k}] \mathcal{R}(\mathrm{d}u_1) \dots \mathcal{R}(\mathrm{d}u_{d-k}).$$

Proof. The result comes directly from Equation (47). $\square$

Corollary 4.1.7. *Let $i = 1, 2, \dots, d-k$. If $0 < s_1 < \dots < s_{i-1} < t$ then $\beta_i(\mathsf{MP}_{k,k}^{(t)})$ is conditionally independent of $\beta_{i-2}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-2}, \dots, \beta_1(\mathsf{MP}_{k,k}^{(t)}) = s_1$ given $\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-1}$.*

Proof. We need to prove that if $0 < s_1 < \dots < s_i < t$ then

$$p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}) | \beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-1}}(s_i) = p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}) | \beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-1}, \beta_{i-2}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-2}, \dots, \beta_1(\mathsf{MP}_{k,k}^{(t)}) = s_1}(s_i).$$

Here $p_{\beta_i(\mathsf{MP}_{k,k}^{(t)})|\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})=s_{i-1}}$ is the conditional density of $\beta_i(\mathsf{MP}_{k,k}^{(t)})$ given that $\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-1}$ and $p_{\beta_i(\mathsf{MP}_{k,k}^{(t)})|\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})=s_{i-1},\beta_{i-2}(\mathsf{MP}_{k,k}^{(t)})=s_{i-2},\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})=s_1}$ is the conditional density of $\beta_i(\mathsf{MP}_{k,k}^{(t)})$ given that $\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-1}, \beta_{i-2}(\mathsf{MP}_{k,k}^{(t)}) = s_{i-2}$, $\dots, \beta_1(\mathsf{MP}_{k,k}^{(t)}) = s_1$. Indeed, we have

$$p_{\beta_i(\mathsf{MP}_{k,k}^{(t)})|\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})=s_{i-1}}(s_i) = \frac{p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}),\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})}(s_i, s_{i-1})}{p_{\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})}(s_{i-1})},$$

where, using Theorem 4.1.1,

$$\begin{aligned} p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}),\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})}(s_i, s_{i-1}) &= \int_{s_i}^{t} \dots \int_{s_{d-k-1}}^{t} \int_{0}^{s_{i-1}} \dots \int_{0}^{s_2} \frac{(d-k)!}{t^{d-k}} \mathrm{d}s_1 \dots \mathrm{d}s_{i-2} \mathrm{d}s_{d-k} \dots \mathrm{d}s_{i+1} \\ &= \frac{(d-k)!}{t^{d-k}} \frac{s_{i-1}^{i-2}}{(i-2)!} \frac{(t-s_i)^{d-k-i}}{(d-k-i)!} \end{aligned}$$

and

$$\begin{aligned} p_{\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})}(s_{i-1}) &= \int_{s_{i-1}}^{t} p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}),\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})}(s_i, s_{i-1}) \mathrm{d}s_i \\ &= \frac{(d-k)!}{t^{d-k}} \frac{s_{i-1}^{i-2}}{(i-2)!} \frac{(t-s_{i-1})^{d-k-i+1}}{(d-k-i+1)!}. \end{aligned}$$

On the other hand,

$$p_{\beta_i(\mathsf{MP}_{k,k}^{(t)})|\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})=s_{i-1},\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})=s_1}(s_i) = \frac{p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})}(s_i, \dots, s_1)}{p_{\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})}(s_{i-1}, \dots, s_1)},$$

where

$$\begin{aligned} p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})}(s_i, \dots, s_1) &= \int_{s_i}^{t} \dots \int_{s_{d-k-1}}^{t} \frac{(d-k)!}{t^{d-k}} \mathrm{d}s_{d-k} \dots \mathrm{d}s_{i+1} \\ &= \frac{(d-k)!}{t^{d-k}} \frac{(t-s_i)^{d-k-i}}{(d-k-i)!} \end{aligned}$$

and

$$\begin{aligned} p_{\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})}(s_{i-1}, \dots, s_1) &= \int_{s_{i-1}}^{t} p_{\beta_i(\mathsf{MP}_{k,k}^{(t)}),\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})}(s_i, \dots, s_1) \mathrm{d}s_i \\ &= \frac{(d-k)!}{t^{d-k}} \frac{(t-s_{i-1})^{d-k-i+1}}{(d-k-i+1)!}. \end{aligned}$$

We conclude that

$$p_{\beta_i(\mathsf{MP}_{k,k}^{(t)})|\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})=s_{i-1}}(s_i) = p_{\beta_i(\mathsf{MP}_{k,k}^{(t)})|\beta_{i-1}(\mathsf{MP}_{k,k}^{(t)})=s_{i-1},\beta_{i-2}(\mathsf{MP}_{k,k}^{(t)})=s_{i-2},\dots,\beta_1(\mathsf{MP}_{k,k}^{(t)})=s_1}(s_i)$$

$$= \frac{(d-k-i+1)(t-s_i)^{d-k-i}}{(t-s_{i-1})^{d-k-i+1}}.$$

□

Lemma 4.1.8. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$, $f : \mathscr{P}_k^o \to \mathbb{R}$ be non-negative and measurable and $t > 0$. Then for almost all $0 < s_1 < \dots < s_{d-k} < t$ we have*

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})|\beta_1 = s_1, \dots, \beta_{d-k} = s_{d-k}\big] = \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})|\beta_{d-k} = s_{d-k}\big]$$

Here $(\beta_1, \dots, \beta_{d-k}) := (\beta_1(\mathsf{MP}_{k,k}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}))$.

Before presenting the proof of Lemma 4.1.8, let $\mathscr{P}_{\leq k}$ equipped with the Hausdorff metric be the measurable space of polytopes with dimension at most k in $\mathbb{R}^d$. Furthermore, put

$$M := \{(\mathsf{p}_1, \dots, \mathsf{p}_{d-k}) \in \mathscr{M}\mathscr{P}_{d-1}^{(t)} \times \dots \times \mathscr{M}\mathscr{P}_{d-1}^{(t)} : \dim(\mathsf{p}_1 \cap \dots \cap \mathsf{p}_{d-k}) = k\}.$$

We use the following proposition in [**20**] for the proof of Lemma 4.1.8.

Proposition 4.1.9. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$, $g : \mathscr{P}_{\leq k} \times (0,t)^{d-k} \to \mathbb{R}$ be non-negative and measurable and $t > 0$. Then*

$$\mathbb{E} \sum_{(\mathsf{p}_1,\beta_1),\dots,(\mathsf{p}_{d-k},\beta_{d-k}) \in \widehat{\mathscr{M}\mathscr{P}}_{d-1}^{(t)}} g\Big(\bigcap_{i=1}^{d-k} \mathsf{p}_i, \beta_1, \dots, \beta_{d-k}\Big) \mathbf{1}_M(\mathsf{p}_1, \dots, \mathsf{p}_{d-k}) \mathbf{1}_{\Delta(t)}((\beta_1, \dots, \beta_{d-k}))$$

$$= \int_0^t \dots \int_0^t \int_{A(d,d-1)} \dots \int_{A(d,d-1)} \mathbb{E} \sum_{\mathsf{z}_{d-k} \in Y(\beta_{d-k})} g(\mathsf{z}_{d-k} \cap \bigcap_{i=1}^{d-k} H_i, \beta_1, \dots, \beta_{d-k}) \times$$

$$\times \mathbf{1}\Big\{ \dim\big(\mathsf{z}_{d-k} \cap \bigcap_{i=1}^{d-k} H_i\big) = k \Big\} \mathbf{1}_{\Delta(t)}((\beta_1, \dots, \beta_{d-k})) \Lambda(\mathrm{d}H_1) \dots \Lambda(\mathrm{d}H_{d-k}) \mathrm{d}\beta_1 \dots \mathrm{d}\beta_{d-k}.$$

Proof of Lemma 4.1.8. For $\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}$, without danger of confusion, we also use the notation $(\beta_1, \dots, \beta_{d-k})$ for $(\beta_1(\mathsf{p}), \dots, \beta_{d-k}(\mathsf{p}))$. Definition 1.4.14 gives us

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) \mathbf{1}_{(B_1 \times \dots \times B_{d-k}) \cap \Delta(t)}((\beta_1, \dots, \beta_{d-k}))\big] = \Big[\mathbb{E} \sum_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p})) V_k(\mathsf{p})\Big]^{-1} \times$$

$$\times \mathbb{E} \sum_{(c(\mathsf{p}),\mathsf{p}_o,\beta_1,\dots,\beta_{d-k}) \in \widetilde{\mathscr{M}\mathscr{P}}_k^{(t)},\, c(\mathsf{p}) \in B} f(\mathsf{p}_o) V_k(\mathsf{p}_o) \mathbf{1}_{\Delta(t)}((\beta_1, \dots, \beta_{d-k})) \mathbf{1}_{B_1}(\beta_1) \dots \mathbf{1}_{B_{d-k}}(\beta_{d-k})$$

$$= \Big[\mathbb{E} \sum_{\mathsf{p} \in \mathscr{M}\mathscr{P}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_k(\mathsf{p})\Big]^{-1} \mathbb{E} \sum_{(\mathsf{p}_1,\beta_1),\dots,(\mathsf{p}_{d-k},\beta_{d-k}) \in \widehat{\mathscr{M}\mathscr{P}}_{d-1}^{(t)}} \mathbf{1}_B\Big(c\big(\bigcap_{i=1}^{d-k} \mathsf{p}_i\big)\Big) V_k\Big(\bigcap_{i=1}^{d-k} \mathsf{p}_i\Big) \times$$

$$\times f\Big(\bigcap_{i=1}^{d-k} \mathsf{p}_i - c\big(\bigcap_{i=1}^{d-k} \mathsf{p}_i\big)\Big) \mathbf{1}_M(\mathsf{p}_1, \dots, \mathsf{p}_{d-k}) \mathbf{1}_{\Delta(t)}((\beta_1, \dots, \beta_{d-k})) \mathbf{1}_{B_1}(\beta_1) \dots \mathbf{1}_{B_{d-k}}(\beta_{d-k})$$

where A is a Borel subset of $\mathscr{P}_k^o$, B is a Borel subset of $\mathbb{R}^d$ with $0 < \lambda_d(B) < \infty$ and $B_1, \ldots, B_{d-k}$ are Borel subsets of $(0,t)$. If the function g in Proposition 4.1.9 is given by

$$g(\mathsf{p}, \beta_1(\mathsf{p}), \ldots, \beta_{d-k}(\mathsf{p})) = \mathbf{1}_B(c(\mathsf{p}))f(\mathsf{p} - c(\mathsf{p}))V_k(\mathsf{p})\mathbf{1}_{B_1}(\beta_1(\mathsf{p}))\ldots\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{p}))$$

then we apply Proposition 4.1.9 and get

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})\mathbf{1}_{(B_1\times\ldots\times B_{d-k})\cap\Delta(t)}((\beta_1,\ldots,\beta_{d-k}))\big] = \Big[\mathbb{E}\sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}}\mathbf{1}_B(c(\mathsf{p}))\,V_k(\mathsf{p})\Big]^{-1}\int\limits_{B_{d-k}}\int\limits_{A(d,d-1)}$$

$$\ldots\int\limits_{A(d,d-1)}\mathbb{E}\sum_{\mathsf{z}_{d-k}\in Y(\beta_{d-k})}\mathbf{1}_B\Big(c\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big)\Big)f\Big(\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big)_o\Big)V_k\Big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\Big)\mathbf{1}\{C\}$$

$$\int\limits_{B_{d-k-1}}\ldots\int\limits_{B_1}\mathbf{1}\{0<\beta_1<\ldots<\beta_{d-k-1}<\beta_{d-k}\}\mathrm{d}\beta_1\ldots\mathrm{d}\beta_{d-k-1}\Lambda(\mathrm{d}H_1)\ldots\Lambda(\mathrm{d}H_{d-k})\mathrm{d}\beta_{d-k},$$

where $\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big)_o := \big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big) - c\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big)$ and $\mathbf{1}\{C\} := \mathbf{1}\{\dim\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big) = k\}$. On the other hand, for fixed $0<\beta_{d-k}<t$, put $\Delta(\beta_{d-k}) := \{(r_1,\ldots,r_{d-k-1})\in\mathbb{R}^{d-k-1} : 0<r_1<\ldots<r_{d-k-1}<\beta_{d-k}\}$. Then $\Delta(\beta_{d-k})$ is a $(d-k-1)$-simplex which is a subset of $\mathbb{R}^{d-k-1}$. We have

$$\int\limits_{B_{d-k-1}}\ldots\int\limits_{B_1}\mathbf{1}\{0<\beta_1<\ldots<\beta_{d-k-1}<\beta_{d-k}\}\mathrm{d}\beta_1\ldots\mathrm{d}\beta_{d-k-1}$$

$$= \lambda_{d-k-1}((B_1\times\ldots\times B_{d-k-1})\cap\Delta(\beta_{d-k})) =: S(\beta_{d-k}),$$

where λ_{d-k-1} is the Lebesgue measure on $\mathbb{R}^{d-k-1}$. Now put

$$h\Big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\Big) := \mathbf{1}_B\Big(c\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big)\Big)f\Big(\big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\big)_o\Big)V_k\Big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\Big)\mathbf{1}\{C\}$$

then

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})\mathbf{1}_{(B_1\times\ldots\times B_{d-k})\cap\Delta(t)}((\beta_1,\ldots,\beta_{d-k}))\big] = \Big[\mathbb{E}\sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}}\mathbf{1}_B(c(\mathsf{p}))\,V_k(\mathsf{p})\Big]^{-1}\int\limits_{B_{d-k}}$$

$$\int\limits_{A(d,d-1)}\ldots\int\limits_{A(d,d-1)}\mathbb{E}\sum_{\mathsf{z}_{d-k}\in Y(\beta_{d-k})}h\Big(\mathsf{z}_{d-k}\cap\bigcap_{i=1}^{d-k}H_i\Big)\Lambda(\mathrm{d}H_1)\ldots\Lambda(\mathrm{d}H_{d-k})S(\beta_{d-k})\mathrm{d}\beta_{d-k}$$

$$= (d-k-1)!\Big[\mathbb{E}\sum_{\mathsf{p}\in\mathscr{M}\mathscr{P}_k^{(t)}}\mathbf{1}_B(c(\mathsf{p}))\,V_k(\mathsf{p})\Big]^{-1}\times$$

$$\int_0^t \int_0^t \dots \int_0^t \int_{A(d,d-1)} \dots \int_{A(d,d-1)} \mathbb{E} \sum_{\mathsf{z}_{d-k} \in Y(\beta_{d-k})} h\Big(\mathsf{z}_{d-k} \cap \bigcap_{i=1}^{d-k} H_i\Big) \Lambda(\mathrm{d}H_1) \dots \Lambda(\mathrm{d}H_{d-k})$$
$$\mathbf{1}_{\Delta(\beta_{d-k})}((\beta_1, \dots, \beta_{d-k-1}))\mathrm{d}\beta_1 \dots \mathrm{d}\beta_{d-k-1} S(\beta_{d-k}) \mathbf{1}_{B_{d-k}}(\beta_{d-k})(\beta_{d-k}^{d-k-1})^{-1}\mathrm{d}\beta_{d-k}.$$

Here we have used, for fixed $0 < \beta_{d-k} < t$,

$$\int_0^t \dots \int_0^t \mathbf{1}_{\Delta(\beta_{d-k})}((\beta_1, \dots, \beta_{d-k-1}))\mathrm{d}\beta_1 \dots \mathrm{d}\beta_{d-k-1} = \lambda_{d-k-1}(\Delta(\beta_{d-k})) = \frac{\beta_{d-k}^{d-k-1}}{(d-k-1)!}.$$

Applying Proposition 4.1.9 we get

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})\mathbf{1}_{(B_1 \times \dots \times B_{d-k}) \cap \Delta(t)}((\beta_1, \dots, \beta_{d-k}))\big] = (d-k-1)! \Big[\mathbb{E} \sum_{\mathsf{p} \in \mathscr{MP}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p}))\, V_k(\mathsf{p})\Big]^{-1} \times$$
$$\times \mathbb{E} \sum_{(\mathsf{p}_1, \beta_1), \dots, (\mathsf{p}_{d-k}, \beta_{d-k}) \in \widehat{\mathscr{MP}}_{d-1}^{(t)}} \mathbf{1}_B\Big(c\big(\bigcap_{i=1}^{d-k} \mathsf{p}_i\big)\Big) f\Big(\bigcap_{i=1}^{d-k} \mathsf{p}_i - c\big(\bigcap_{i=1}^{d-k} \mathsf{p}_i\big)\Big) V_k\Big(\bigcap_{i=1}^{d-k} \mathsf{p}_i\Big) \times$$
$$\times S(\beta_{d-k}) \mathbf{1}_{B_{d-k}}(\beta_{d-k})(\beta_{d-k}^{d-k-1})^{-1} \mathbf{1}_M(\mathsf{p}_1, \dots, \mathsf{p}_{d-k}) \mathbf{1}_{\Delta(t)}((\beta_1, \dots, \beta_{d-k}))$$
$$= (d-k-1)! \times \mathbb{E} \sum_{(c(\mathsf{p}), \mathsf{p}_o, \beta_{d-k}) \in \widehat{\mathscr{MP}}_k^{(t)}} \mathbf{1}_B(c(\mathsf{p})) f(\mathsf{p}_o) V_k(\mathsf{p}_o) S(\beta_{d-k}) \mathbf{1}_{B_{d-k}}(\beta_{d-k})(\beta_{d-k}^{d-k-1})^{-1}$$
$$= (d-k-1)! \times \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) S(\beta_{d-k}) \mathbf{1}_{B_{d-k}}(\beta_{d-k})(\beta_{d-k}^{d-k-1})^{-1}\big].$$

Using Theorem 4.1.1, we find that

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})\mathbf{1}_{(B_1 \times \dots \times B_{d-k}) \cap \Delta(t)}((\beta_1, \dots, \beta_{d-k}))\big] = \frac{(d-k)!}{t^{d-k}} \int_{B_{d-k}} \dots \int_{B_1}$$
$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) | \beta_1 = s_1, \dots, \beta_{d-k} = s_{d-k}\big] \mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\} \mathrm{d}s_1 \dots \mathrm{d}s_{d-k},$$

whereas using Remark 4.1.5, we obtain

$$(d-k-1)! \times \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) S(\beta_{d-k}) \mathbf{1}_{B_{d-k}}(\beta_{d-k})(\beta_{d-k}^{d-k-1})^{-1}\big] = (d-k-1)! \int_{B_{d-k}}$$
$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) | \beta_{d-k} = s_{d-k}\big] S(s_{d-k})(s_{d-k}^{d-k-1})^{-1}(d-k) s_{d-k}^{d-k-1} t^{-(d-k)} \mathrm{d}s_{d-k}$$
$$= \frac{(d-k)!}{t^{d-k}} \int_{B_{d-k}} \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) | \beta_{d-k} = s_{d-k}\big] \int_{B_{d-k-1}} \dots \int_{B_1} \mathbf{1}\{0 < s_1 < \dots < s_{d-k-1} < s_{d-k}\}$$
$$\mathrm{d}s_1 \dots \mathrm{d}s_{d-k-1} \mathrm{d}s_{d-k}$$
$$= \frac{(d-k)!}{t^{d-k}} \int_{B_{d-k}} \dots \int_{B_1} \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)}) | \beta_{d-k} = s_{d-k}\big] \mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\} \mathrm{d}s_1 \dots \mathrm{d}s_{d-k}.$$

This leads to

$$\int\limits_{B_{d-k}} \dots \int\limits_{B_1} \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})|\beta_1 = s_1, \dots, \beta_{d-k} = s_{d-k}\big]\mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\}\mathrm{d}s_1 \dots \mathrm{d}s_{d-k}$$

$$= \int\limits_{B_{d-k}} \dots \int\limits_{B_1} \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})|\beta_{d-k} = s_{d-k}\big]\mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\}\mathrm{d}s_1 \dots \mathrm{d}s_{d-k}$$

for all Borel subsets $B_1, \dots, B_{d-k}$ of $(0,t)$. This completes our proof. $\square$

Lemma 4.1.10. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$, $f : \mathscr{P}_k^o \to \mathbb{R}$ be non-negative and measurable and $t > 0$. Then*

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})|\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) = s_{d-k}\big] = \mathbb{E}\big[f(\mathsf{F}_{k,k}^{(s_{d-k})})\big]$$

for almost all $0 < s_{d-k} < t$.

Proof. For any $B_{d-k} \in \mathcal{B}((0,t))$, Lemma 3.6.1 for the case $j = k$ gives us

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}))\big] = \int\limits_{B_{d-k}} \frac{(d-k)s_{d-k}^{d-k-1}}{t^{d-k}}\,\mathbb{E}f(\mathsf{F}_{k,k}^{(s_{d-k})})\,\mathrm{d}s_{d-k}$$

On the other hand, using Remark 4.1.5, we find that

$$\mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}))\big]$$

$$= \int\limits_{B_{d-k}} \mathbb{E}\big[f(\mathsf{MP}_{k,k}^{(t)})|\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) = s_{d-k}\big]\frac{(d-k)s_{d-k}^{d-k-1}}{t^{d-k}}\mathrm{d}s_{d-k}$$

and our assertion follows. $\square$

4.2. Birth-time distributions of V_j-weighted polytopes

Theorem 4.2.1. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$ and $j \in \{0, \dots, k\}$. The joint distribution of the birth-times $\beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)})$ of the V_j-weighted typical k-dimensional maximal polytope $\mathsf{MP}_{k,j}^{(t)}$ of the STIT tessellation $Y(t)$ has density*

$$(s_1, \dots, s_{d-k}) \mapsto (d-j)(d-k-1)!\frac{s_{d-k}^{k-j}}{t^{d-j}}\,\mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\}$$

with respect to the Lebesgue measure on the $(d-k)$-dimensional simplex $\Delta(t)$, which is independent of the hyperplane measure Λ. In particular, if $j = k$ we obtain the uniform distribution on $\Delta(t)$ as in Theorem 4.1.1.

Proof. According to Proposition 1.4.15, we get

$$\mathbb{E}[f(\mathsf{MP}_{k,j}^{(t)}, \beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\mathbf{1}_{\Delta(t)}((\beta_1(\mathsf{MP}_{k,j}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)})))]$$

$$= \frac{\mathbb{E}V_k(\mathsf{MP}_{k,0}^{(t)})}{\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})} \times \mathbb{E}[f(\mathsf{MP}_{k,k}^{(t)}, \beta_1(\mathsf{MP}_{k,k}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}))\times$$

$$\times\,\mathbf{1}_{\Delta(t)}((\beta_1(\mathsf{MP}_{k,k}^{(t)}), \dots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})))V_j(\mathsf{MP}_{k,k}^{(t)})V_k(\mathsf{MP}_{k,k}^{(t)})^{-1}]. \tag{48}$$

for any non-negative measurable function f on $\mathscr{P}_k^o \times (0,t)^{d-k}$. Let us fix $0 < r_1 < \ldots < r_{d-k} < t$ and apply Equation (48) for the function $f : \mathscr{P}_k^o \times (0,t)^{d-k} \to \mathbb{R}$ given by

$$f(p, \beta_1(p), \ldots, \beta_{d-k}(p)) = \mathbf{1}\big\{\beta_1(p) \in (0, r_1), \ldots, \beta_{d-k}(p) \in (r_{d-k-1}, r_{d-k})\big\}$$

to obtain

$$\begin{aligned}
&\mathbb{P}\big(\beta_1(\mathsf{MP}_{k,j}^{(t)}) \in (0,r_1), \ldots, \beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}) \in (r_{d-k-1}, r_{d-k})\big)\\
&= \frac{\mathbb{E}V_k(\mathsf{MP}_{k,0}^{(t)})}{\mathbb{E}V_j(\mathsf{MP}_{k,0}^{(t)})}\, \mathbb{E}\big[\mathbf{1}\{\beta_1(\mathsf{MP}_{k,k}^{(t)}) \in (0,r_1), \ldots, \\
&\qquad\qquad \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) \in (r_{d-k-1}, r_{d-k})\}\, V_j(\mathsf{MP}_{k,k}^{(t)})\, V_k(\mathsf{MP}_{k,k}^{(t)})^{-1}\big].
\end{aligned} \tag{49}$$

Conditioning on the birth-times and using Theorem 4.1.1 yield

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}\{\beta_1(\mathsf{MP}_{k,k}^{(t)}) \in (0,r_1), \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) \in (r_{d-k-1}, r_{d-k})\}\, V_j(\mathsf{MP}_{k,k}^{(t)})\, V_k(\mathsf{MP}_{k,k}^{(t)})^{-1}\big]\\
&= \int_{r_{d-k-1}}^{r_{d-k}} \cdots \int_{r_1}^{r_2} \int_0^{r_1} \mathbb{E}\big[V_j(\mathsf{MP}_{k,k}^{(t)})\, V_k(\mathsf{MP}_{k,k}^{(t)})^{-1} | \beta_1(\mathsf{MP}_{k,k}^{(t)}) = s_1, \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) = s_{d-k}\big] \times \\
&\qquad\qquad \times \frac{(d-k)!}{t^{d-k}} \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_{d-k}.
\end{aligned}$$

Lemma 4.1.8 and Lemma 4.1.10 imply that the joint conditional distribution of $(V_k(\mathsf{MP}_{k,k}^{(t)}), V_j(\mathsf{MP}_{k,k}^{(t)}))$ of the k-volume-weighted typical k-maximal polytope $\mathsf{MP}_{k,k}^{(t)}$ of $Y(t)$, given its birth-times $(\beta_1(\mathsf{MP}_{k,k}^{(t)}), \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)})) = (s_1, \ldots, s_{d-k})$, only depends on the last birth-time $\beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) = s_{d-k}$ and equals the joint distribution of $(V_k(\mathsf{F}_{k,k}^{(s_{d-k})}), V_j(\mathsf{F}_{k,k}^{(s_{d-k})}))$ of the k-volume-weighted typical k-face $\mathsf{F}_{k,k}^{(s_{d-k})}$ in $\mathrm{PHT}(s_{d-k}\Lambda)$. Whence, due to the scaling property (38) and the homogeneity of the intrinsic volume V_j we infer that

$$\begin{aligned}
&\mathbb{E}\big[V_j(\mathsf{MP}_{k,k}^{(t)})\, V_k(\mathsf{MP}_{k,k}^{(t)})^{-1} | \beta_1(\mathsf{MP}_{k,k}^{(t)}) = s_1, \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) = s_{d-k}\big]\\
&= \mathbb{E}\big[V_j(\mathsf{F}_{k,k}^{(s_{d-k})})\, V_k(\mathsf{F}_{k,k}^{(s_{d-k})})^{-1}\big] = \mathbb{E}\big[V_j(s_{d-k}^{-1}\mathsf{F}_{k,k}^{(1)})\, V_k(s_{d-k}^{-1}\mathsf{F}_{k,k}^{(1)})^{-1}\big]\\
&= s_{d-k}^{k-j}\, \mathbb{E}\big[V_j(\mathsf{F}_{k,k}^{(1)})\, V_k(\mathsf{F}_{k,k}^{(1)})^{-1}\big] = c_1 s_{d-k}^{k-j},
\end{aligned}$$

where $c_1 := \mathbb{E}\big[V_j(\mathsf{F}_{k,k}^{(1)})\, V_k(\mathsf{F}_{k,k}^{(1)})^{-1}\big]$ is a constant only depending on j, k, d and Λ. So,

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}\{\beta_1(\mathsf{MP}_{k,k}^{(t)}) \in (0,r_1), \ldots, \beta_{d-k}(\mathsf{MP}_{k,k}^{(t)}) \in (r_{d-k-1}, r_{d-k})\} V_j(\mathsf{MP}_{k,k}^{(t)})\, V_k(\mathsf{MP}_{k,k}^{(t)})^{-1}\big]\\
&\qquad = \int_{r_{d-k-1}}^{r_{d-k}} \cdots \int_{r_1}^{r_2} \int_0^{r_1} c_1 s_{d-k}^{k-j}\, \frac{(d-k)!}{t^{d-k}}\, \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_{d-k}.
\end{aligned} \tag{50}$$

Moreover, according to Lemma 1.4.17b), there is another constant c_2, only depending on the dimension parameters j, k, d and on the hyperplane measure Λ, such that

$$\frac{\mathbb{E}V_k(\mathsf{MP}^{(t)}_{k,0})}{\mathbb{E}V_j(\mathsf{MP}^{(t)}_{k,0})} = c_2\, t^{j-k}. \tag{51}$$

Putting $c_3 := c_1 c_2$ and combining Equation (49) with Equation (50) and Equation (51), we arrive at

$$\begin{aligned}&\mathbb{P}\big(\beta_1(\mathsf{MP}^{(t)}_{k,j}) \in (0, r_1), \dots, \beta_{d-k}(\mathsf{MP}^{(t)}_{k,j}) \in (r_{d-k-1}, r_{d-k})\big)\\ &= \int\limits_{r_{d-k-1}}^{r_{d-k}} \dots \int\limits_{r_1}^{r_2} \int\limits_{0}^{r_1} c_3\,(d-k)\,! \frac{s_{d-k}^{k-j}}{t^{d-j}}\, \mathrm{d}s_1 \mathrm{d}s_2 \dots \mathrm{d}s_{d-k}.\end{aligned}$$

Similar to the argument in the proof of Theorem 4.1.1, only the derivative of $\mathbb{P}\big(\beta_1 \in (0, r_1), \beta_2 \in (r_1, r_2), \dots, \beta_{d-k} \in (r_{d-k-1}, r_{d-k})\big)$ remains after differentiation of $\mathbb{P}\big(\beta_1 \in (0, r_1), \beta_2 \in (0, r_2), \dots, \beta_{d-k} \in (0, r_{d-k})\big)$ with respect to $r_1, \dots, r_{d-k}$. This shows that the birth-times of $\mathsf{MP}^{(t)}_{k,j}$ have a joint density given by

$$(r_1, \dots, r_{d-k}) \mapsto c_3\,(d-k)!\, \frac{r_{d-k}^{k-j}}{t^{d-j}} \mathbf{1}\{0 < r_1 < \dots < r_{d-k} < t\}.$$

Since this must integrate to 1, we must have $c_3 = (d-j)/(d-k)$. This finally proves that

$$(s_1, \dots, s_{d-k}) \mapsto (d-j)(d-k-1)!\, \frac{s_{d-k}^{k-j}}{t^{d-j}} \mathbf{1}\{0 < s_1 < \dots < s_{d-k} < t\}$$

is the joint birth-time density of $\mathsf{MP}^{(t)}_{k,j}$. □

Remark 4.2.2. In the special cases $d = 2$ or $d = 3$, $k = 1$ and $j = 0$ or $j = 1$, the formula in Theorem 4.2.1 is known from [**33, 16, 36**].

Remark 4.2.3. It is worth pointing out that the density

$$s_{d-k} \mapsto (d-j) s_{d-k}^{d-j-1} t^{-(d-j)} \mathbf{1}\{0 < s_{d-k} < t\}$$

is the marginal density of the last birth-time $\beta_{d-k}(\mathsf{MP}^{(t)}_{k,j})$ of the V_j-weighted typical maximal polytope of dimension k in Theorem 4.2.1.

Corollary 4.2.4. *Let $d \geq 2$, $k \in \{0, \dots, d-1\}$, $j \in \{0, \dots, k\}$, $f : \mathscr{P}_k^o \to \mathbb{R}$ be non-negative and measurable and $t > 0$. Then for almost all $0 < s_1 < \dots < s_{d-k} < t$ we have*

$$\mathbb{E}\big[f(\mathsf{MP}^{(t)}_{k,j}) | \beta_1 = s_1, \dots, \beta_{d-k} = s_{d-k}\big] = \mathbb{E}\big[f(\mathsf{MP}^{(t)}_{k,j}) | \beta_{d-k} = s_{d-k}\big]$$

Here $(\beta_1, \dots, \beta_{d-k}) := (\beta_1(\mathsf{MP}^{(t)}_{k,j}), \dots, \beta_{d-k}(\mathsf{MP}^{(t)}_{k,j}))$.

Proof. Similar to Proof of Lemma 4.1.8, we have, for $B_1, \dots, B_{d-k} \in \mathcal{B}((0,t))$,

$$\begin{aligned}&\mathbb{E}\big[f(\mathsf{MP}^{(t)}_{k,j}) \mathbf{1}_{(B_1 \times \dots \times B_{d-k}) \cap \Delta(t)}((\beta_1, \dots, \beta_{d-k}))\big]\\ &= (d-k-1)! \times \mathbb{E}\big[f(\mathsf{MP}^{(t)}_{k,j}) S(\beta_{d-k}) \mathbf{1}_{B_{d-k}}(\beta_{d-k}) (\beta_{d-k}^{d-k-1})^{-1}\big].\end{aligned}$$

Using Theorem 4.2.1, we find that

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})\mathbf{1}_{(B_1\times\ldots\times B_{d-k})\cap\Delta(t)}((\beta_1,\ldots,\beta_{d-k}))\big] = \frac{(d-j)(d-k-1)!}{t^{d-j}}\int\limits_{B_{d-k}}\cdots\int\limits_{B_1}$$

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_1=s_1,\ldots,\beta_{d-k}=s_{d-k}\big]\mathbf{1}\{0<s_1<\ldots<s_{d-k}<t\}\,s_{d-k}^{k-j}\mathrm{d}s_1\ldots\mathrm{d}s_{d-k},$$

whereas using Remark 4.2.3, we obtain

$$\begin{aligned}
&(d-k-1)!\times\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})S(\beta_{d-k})\mathbf{1}_{B_{d-k}}(\beta_{d-k})(\beta_{d-k}^{d-k-1})^{-1}\big]\\
&=(d-k-1)!\int\limits_{B_{d-k}}\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_{d-k}=s_{d-k}\big]S(s_{d-k})(s_{d-k}^{d-k-1})^{-1}(d-j)s_{d-k}^{d-j-1}t^{-(d-j)}\mathrm{d}s_{d-k}\\
&=\frac{(d-j)(d-k-1)!}{t^{d-j}}\int\limits_{B_{d-k}}\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_{d-k}=s_{d-k}\big]\int\limits_{B_{d-k-1}}\cdots\int\limits_{B_1}\\
&\qquad\mathbf{1}\{0<s_1<\ldots<s_{d-k-1}<s_{d-k}\}\mathrm{d}s_1\ldots\mathrm{d}s_{d-k-1}s_{d-k}^{k-j}\mathrm{d}s_{d-k}\\
&=\frac{(d-j)(d-k-1)!}{t^{d-j}}\int\limits_{B_{d-k}}\cdots\int\limits_{B_1}\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_{d-k}=s_{d-k}\big]\mathbf{1}\{0<s_1<\ldots<s_{d-k}<t\}\\
&\qquad s_{d-k}^{k-j}\mathrm{d}s_1\ldots\mathrm{d}s_{d-k}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\int\limits_{B_{d-k}}\cdots\int\limits_{B_1}\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_1=s_1,\ldots,\beta_{d-k}=s_{d-k}\big]\mathbf{1}\{0<s_1<\ldots<s_{d-k}<t\}\\
&\qquad s_{d-k}^{k-j}\mathrm{d}s_1\ldots\mathrm{d}s_{d-k}\\
&=\int\limits_{B_{d-k}}\cdots\int\limits_{B_1}\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_{d-k}=s_{d-k}\big]\mathbf{1}\{0<s_1<\ldots<s_{d-k}<t\}s_{d-k}^{k-j}\mathrm{d}s_1\ldots\mathrm{d}s_{d-k}
\end{aligned}$$

for all Borel subsets $B_1,\ldots,B_{d-k}$ of $(0,t)$. This completes our proof. $\square$

Corollary 4.2.5. *Let $d\geq 2$, $k\in\{0,\ldots,d-1\}$, $j\in\{0,\ldots,k\}$, $f:\mathscr{P}_k^o\to\mathbb{R}$ be non-negative and measurable and $t>0$. Then*

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)})=s_{d-k}\big]=\mathbb{E}\big[f(\mathsf{F}_{k,j}^{(s_{d-k})})\big]$$

for almost all $0<s_{d-k}<t$.

Proof. For any $B_{d-k}\in\mathcal{B}((0,t))$, Lemma 3.6.1 gives us

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\big]=\int\limits_{B_{d-k}}\frac{(d-j)s_{d-k}^{d-j-1}}{t^{d-j}}\,\mathbb{E}f(\mathsf{F}_{k,j}^{(s_{d-k})})\,\mathrm{d}s_{d-k}$$

On the other hand, using Remark 4.2.3, we find that

$$\mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})\,\mathbf{1}_{B_{d-k}}(\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}))\big]=$$

$$= \int\limits_{B_{d-k}} \mathbb{E}\big[f(\mathsf{MP}_{k,j}^{(t)})|\beta_{d-k}(\mathsf{MP}_{k,j}^{(t)}) = s_{d-k}\big]\frac{(d-j)s_{d-k}^{d-j-1}}{t^{d-j}}\mathrm{d}s_{d-k}$$

and our assertion follows. □

4.3. Applications of Theorem 4.2.1

We turn now to an application of Theorem 4.2.1, where we consider the typical and the length-weighted typical maximal segment $\mathsf{MP}_{1,0}^{(t)}$ and $\mathsf{MP}_{1,1}^{(t)}$ of $Y(t)$, respectively. These segments may have internal vertices, which arise at the time of birth of the segment (when $d \geq 3$) and thereafter subject to further subdivision of adjacent cells. For example in the planar case, a maximal segment is always born without internal vertices. However, after the time of birth, vertices in its relative interior can be created by the intersection with other maximal segments that are born later to the left or to the right of the segment; the bold maximal segment shown in Figure 12 provides an illustration of that phenomenon. With the help of Theorem 4.2.1 we can determine the probabilities $\mathsf{p}_{1,0}(n)$ and $\mathsf{p}_{1,1}(n)$ that the typical or the length-weighted typical maximal segment of $Y(t)$ contains exactly $n \in \{0, 1, 2, \dots\}$ internal vertices (we suppress the dependency on t in the notation of these probabilities since they will be shown to be independent of the time parameter t).

Theorem 4.3.1. *The probabilities $\mathsf{p}_{1,0}(n)$ and $\mathsf{p}_{1,1}(n)$ are given by*

$$\mathsf{p}_{1,0}(n) = d(d-2)! \int\limits_0^t \int\limits_0^{s_{d-1}} \dots \int\limits_0^{s_2} \frac{s_{d-1}^2}{t^d} \frac{(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)^n}{(d\cdot t - s_{d-1} - s_{d-2} - \ldots - s_1)^{n+1}} \mathrm{d}s_1 \ldots \mathrm{d}s_{d-2}\mathrm{d}s_{d-1}$$

and $$\mathsf{p}_{1,1}(n) = (n+1)(d-1)! \int\limits_0^t \int\limits_0^{s_{d-1}} \dots \int\limits_0^{s_2} \frac{s_{d-1}^2}{t^{d-1}} \frac{(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)^n}{(d\cdot t - s_{d-1} - s_{d-2} - \ldots - s_1)^{n+2}} \mathrm{d}s_1 \ldots \mathrm{d}s_{d-2}\mathrm{d}s_{d-1}.$$

Moreover, $\mathsf{p}_{1,0}(n)$ and $\mathsf{p}_{1,1}(n)$ are independent of t and Λ. In average, the typical maximal segment has $\frac{1}{2}(d^2 - d + 2)/(d-1)$ internal vertices in dimension $d \geq 2$, whereas the length-weighted typical maximal segment in space dimension $d \geq 3$ has $(d^2 - 2d + 4)/(d-2)$ (the mean is infinite if $d = 2$).

Proof. The proof follows the idea of the corresponding special cases $d = 2$ and $d = 3$ in [**33, 16, 36**]. We will combine the results of Lemma 4.1.8, Lemma 4.1.10 and Theorem 4.2.1.

First, we show the formula for the case of the length-weighted typical maximal segment $\mathsf{MP}_{1,1}^{(t)}$. The birth-time vector of $\mathsf{MP}_{1,1}^{(t)}$ is

$$(\beta_1(\mathsf{MP}_{1,1}^{(t)}), \dots, \beta_{d-1}(\mathsf{MP}_{1,1}^{(t)})) =: (\beta_1, \dots, \beta_{d-1}).$$

Furthermore, let $\ell(\mathsf{MP}_{1,1}^{(t)})$ and $D(\mathsf{MP}_{1,1}^{(t)})$ be the length and the direction of that segment, respectively. Recall that $D : \mathcal{L} \to \mathscr{S}_+^{d-1}$ is a function that assigns to

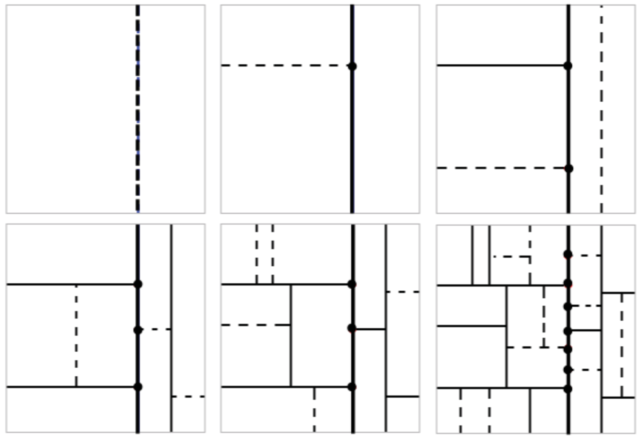

FIGURE 12. The bold maximal segment illustrates the development in time of the internal structure of maximal segments in space dimension two.

$L \in \mathcal{L}$ the unit vector $D(L) \in \mathscr{S}_+^{d-1}$ parallel to L. Without danger of confusion, we shall write ℓ and D instead of $\ell(\mathsf{MP}_{1,1}^{(t)})$ and $D(\mathsf{MP}_{1,1}^{(t)})$, respectively. Let $\mathbb{P}_D$ denote the distribution of D.

Denote by N the number of internal vertices of $\mathsf{MP}_{1,1}^{(t)}$. By conditioning, we find that

$$\mathsf{p}_{1,1}(n) = \mathbb{P}(N = n) =$$

$$= \int_0^t \int_0^{s_{d-1}} \cdots \int_0^{s_2} \int_{\mathscr{S}_+^{d-1}} \int_0^{\infty} \mathbb{P}\big(N = n | \ell = x, D = u, \beta_1 = s_1, \dots, \beta_{d-1} = s_{d-1}\big) \times$$

$$\times p_{\ell,\beta_1,\dots,\beta_{d-1}|D=u}(x, s_1, \dots, s_{d-1}) \, \mathrm{d}x \, \mathbb{P}_D(\mathrm{d}u) \, \mathrm{d}s_1 \dots \mathrm{d}s_{d-2} \mathrm{d}s_{d-1},$$

where $p_{\ell,\beta_1,\dots,\beta_{d-1}|D=u}$ is the conditional joint density of the length ℓ and the birth-times $\beta_1, \dots, \beta_{d-1}$, given that the direction D of the segment equals to u.

We now determine the exact expression for $p_{\ell,\beta_1,\dots,\beta_{d-1}|D=u}$. To start with, write

$$p_{\ell,\beta_1,\dots,\beta_{d-1}|D=u}(x, s_1, \dots, s_{d-1}) = p_{\ell|D=u,\beta_1=s_1,\dots,\beta_{d-1}=s_{d-1}}(x) p_{\beta_1,\dots,\beta_{d-1}|D=u}(s_1, \dots, s_{d-1}) \tag{52}$$

where $p_{\ell|D=u,\beta_1=s_1,\dots,\beta_{d-1}=s_{d-1}}$ is the conditional density of the length ℓ given that $D = u, \beta_1 = s_1, \dots, \beta_{d-1} = s_{d-1}$ and $p_{\beta_1,\dots,\beta_{d-1}|D=u}$ is the conditional joint density of the birth-times $\beta_1, \dots, \beta_{d-1}$ given that the direction D of the segment equals to u.

Denote by $\mathbb{P}_{D|\beta_1=s_1,\dots,\beta_{d-1}=s_{d-1}}$, $\mathbb{P}_{D|\beta_{d-1}=s_{d-1}}$ and $\mathbb{P}_{D(\mathsf{F}_{1,1}^{(s_{d-1})})}$ the conditional distribution of D given that $\beta_1 = s_1, \dots, \beta_{d-1} = s_{d-1}$, the conditional distribution of D given that $\beta_{d-1} = s_{d-1}$, and the distribution of $D(\mathsf{F}_{1,1}^{(s_{d-1})})$, in that order. Then Lemmas 4.1.8 and 4.1.10 give us

$$\mathbb{P}_{D|\beta_1=s_1,\dots,\beta_{d-1}=s_{d-1}} = \mathbb{P}_{D|\beta_{d-1}=s_{d-1}} = \mathbb{P}_{D(\mathsf{F}_{1,1}^{(s_{d-1})})} \tag{53}$$

for almost all $0 < s_1 < \ldots < s_{d-1} < t$. According to [**8**, Theorem 1], the distribution of $D(\mathsf{F}_{1,1}^{(s_{d-1})})$, namely, the directional distribution of the length-weighted typical edge $\mathsf{F}_{1,1}^{(s_{d-1})}$ in $\mathrm{PHT}(s_{d-1}\Lambda)$, is the probability measure $\mathbb{Q}_{d-1}$ on $\mathscr{S}_+^{d-1}$; see Proposition 3.6.6. It is easy to see that, the directional distribution of $\mathsf{F}_{1,1}^{(s_{d-1})}$ is invariant with respect to the scaling factor s_{d-1}. Consequently, the direction $D = D(\mathsf{MP}_{1,1}^{(t)})$ is independent of the birth-time vector $(\beta_1, \ldots, \beta_{d-1})$. We get, using Theorem 4.1.1 for $k = 1$,

$$p_{\beta_1,\ldots,\beta_{d-1}|D=u}(s_1, \ldots, s_{d-1}) = p_{\beta_1,\ldots,\beta_{d-1}}(s_1, \ldots, s_{d-1}) = \frac{(d-1)!}{t^{d-1}}. \tag{54}$$

Moreover, Lemmas 4.1.8 and 4.1.10 show that, for $B \in \mathcal{B}((0,\infty))$ and $U \in \mathcal{B}(\mathscr{S}_+^{d-1})$,

$$\mathbb{P}_{\ell,D|\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(B \times U) = \mathbb{P}_{\ell,D|\beta_{d-1}=s_{d-1}}(B \times U) = \mathbb{P}_{\ell(\mathsf{F}_{1,1}^{(s_{d-1})}),D(\mathsf{F}_{1,1}^{(s_{d-1})})}(B \times U),$$

where

$$\mathbb{P}_{\ell,D|\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(B \times U) = \int\limits_U \int\limits_B p_{\ell|D=u,\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(x)\mathrm{d}x\mathbb{P}_{D|\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(\mathrm{d}u),$$

$$\mathbb{P}_{\ell,D|\beta_{d-1}=s_{d-1}}(B \times U) = \int\limits_U \int\limits_B p_{\ell|D=u,\beta_{d-1}=s_{d-1}}(x)\mathrm{d}x\mathbb{P}_{D|\beta_{d-1}=s_{d-1}}(\mathrm{d}u),$$

and

$$\mathbb{P}_{\ell(\mathsf{F}_{1,1}^{(s_{d-1})}),D(\mathsf{F}_{1,1}^{(s_{d-1})})}(B \times U) = \int\limits_U \int\limits_B p_{\ell(\mathsf{F}_{1,1}^{(s_{d-1})})|D(\mathsf{F}_{1,1}^{(s_{d-1})})=u}(x)\mathrm{d}x\mathbb{P}_{D(\mathsf{F}_{1,1}^{(s_{d-1})})}(\mathrm{d}u).$$

Consequently,

$$\begin{aligned}
&\int\limits_U \int\limits_B p_{\ell|D=u,\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(x)\mathrm{d}x\mathbb{P}_{D|\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(\mathrm{d}u)\\
&= \int\limits_U \int\limits_B p_{\ell|D=u,\beta_{d-1}=s_{d-1}}(x)\mathrm{d}x\mathbb{P}_{D|\beta_{d-1}=s_{d-1}}(\mathrm{d}u)\\
&= \int\limits_U \int\limits_B p_{\ell(\mathsf{F}_{1,1}^{(s_{d-1})})|D(\mathsf{F}_{1,1}^{(s_{d-1})})=u}(x)\mathrm{d}x\mathbb{P}_{D(\mathsf{F}_{1,1}^{(s_{d-1})})}(\mathrm{d}u)
\end{aligned}$$

for any $U \in \mathcal{B}(\mathscr{S}_+^{d-1})$ and for any $B \in \mathcal{B}((0,\infty))$. Using Equation (53), we arrive at

$$p_{\ell|D=u,\beta_1=s_1,\ldots,\beta_{d-1}=s_{d-1}}(x) = p_{\ell|D=u,\beta_{d-1}=s_{d-1}}(x) = p_{\ell(\mathsf{F}_{1,1}^{(s_{d-1})})|D(\mathsf{F}_{1,1}^{(s_{d-1})})=u}(x)$$

for almost all $x \in (0,\infty)$, $u \in \mathscr{S}_+^{d-1}$ and $0 < s_1 < \ldots < s_{d-1} < t$. The last term in the display is the conditional length density of the length-weighted typical edge, given that its direction is u, of the stationary Poisson hyperplane tessellation $\mathrm{PHT}(s_{d-1}\Lambda)$. As shown in Proof of Lemma 3.6.7, the corresponding conditional length distribution is the Gamma (more precisely, Erlang) distribution with parameter $(2, \Lambda(\langle[0,u]\rangle)s_{d-1})$. Hence

$$\begin{aligned} p_{\ell|D=u,\beta_1=s_1,\dots,\beta_{d-1}=s_{d-1}}(x) &= p_{\ell(\mathsf{F}_{1,1}^{(s_{d-1})})|D(\mathsf{F}_{1,1}^{(s_{d-1})})=u}(x) \\ &= [\Lambda(\langle[0,u]\rangle)]^2 s_{d-1}^2 x e^{-\Lambda(\langle[0,u]\rangle)s_{d-1}x}. \end{aligned} \tag{55}$$

Inserting (55) and (54) into (52), we arrive at

$$p_{\ell,\beta_1,\dots,\beta_{d-1}|D=u}(x, s_1, \dots, s_{d-1}) = \frac{(d-1)!}{t^{d-1}}[\Lambda(\langle[0,u]\rangle)]^2 s_{d-1}^2 x e^{-\Lambda(\langle[0,u]\rangle)s_{d-1}x}.$$

It remains to determine the conditional probability

$$\mathbb{P}\big(N = n|\ell = x, D = u, \beta_1 = s_1, \dots, \beta_{d-1} = s_{d-1}\big).$$

Suppose now that the length $\ell = x > 0$, the direction $D = u \in \mathscr{S}_+^{d-1}$ and corresponding birth-times $\beta_1 = s_1, \dots, \beta_{d-1} = s_{d-1}$ with $0 < s_1 < \dots < s_{d-1} < t$ are given. We first notice that for space dimensions $d \geq 3$, a maximal segment can already have internal vertices at the time of its birth. We now gradually reconstruct the internal structure of a maximal segment l with length x and direction u which is the intersection of $(d-1)$ maximal polytopes of dimension $(d-1)$. These polytopes are denoted by $\mathsf{p}_1, \dots, \mathsf{p}_{d-1}$ and have birth-times $s_1, \dots, s_{d-1}$, respectively. Recall that p_1^+ and p_1^- stand for the two closed half-spaces specified by the hyperplane containing p_1. The whole space $\mathbb{R}^d$ is divided into 2 parts, namely, p_1^+ and p_1^-. Assume that at time s_2 the second maximal polytope p_2 is born in p_1^+, which implies that the last $(d-1)$-dimensional maximal polytope p_{d-1} will be also born in p_1^+. Then subdivisions in p_1^+ from time s_1 to time s_2 do not have any influence on the relative interior of l. From time s_1 until time s_2, a number of internal vertices of l appear since p_1^- can be further subdivided during the construction in the time interval (s_1, s_2). This variable is Poisson distributed with parameter $\Lambda(\langle[0,u]\rangle)x(s_2 - s_1)$; see Property [**Linear sections**] in Page 28 of STIT tessellations, for such an l. We infer that the number of internal vertices of $\mathsf{MP}_{1,1}^{(t)}$ appearing from time s_1 until time s_2 is also Poisson distributed with parameter $\Lambda(\langle[0,u]\rangle)x(s_2 - s_1)$. At time s_2 that p_2 is born in p_1^+, $\mathbb{R}^d$ is divided into 3 parts, namely, p_1^-, p_2^- and p_2^+; see the beginning of Proof of Lemma 4.1.2. Then p_3, and consequently, p_{d-1} could only appear in p_2^- or p_2^+. Without loss of generality, assume that p_{d-1} will appear in p_2^+. Hence, internal vertices of l could be only created by further subdivisions within p_1^- and p_2^- from time s_2 until time s_3 (because subdivisions in p_2^+ do not have any influence on the relative interior of l in (s_2, s_3)). Thus, similar to the argument in the time interval (s_1, s_2), for the length-weighted typical maximal segment $\mathsf{MP}_{1,1}^{(t)}$, we obtain in the time interval (s_2, s_3) a Poisson distributed number of internal vertices with parameter $2\Lambda(\langle[0,u]\rangle)x(s_3 - s_2)$. In general, at time s_{j+1}, $1 \leq j \leq d-3$, the $(j+1)$th maximal polytope of dimension $(d-1)$, namely, p_{j+1}, is born in p_j^+ or p_j^- (say p_j^+). Then, $\mathbb{R}^d$ is divided into $(j+2)$ parts which are $\mathsf{p}_1^-, \dots, \mathsf{p}_j^-, \mathsf{p}_{j+1}^-, \mathsf{p}_{j+1}^+$. There are only 2 parts among them, in particular, p_{j+1}^- and p_{j+1}^+ in which p_{d-1} will appear. Assume that p_{d-1} will appear in p_{j+1}^+. Therefore, internal vertices can be created by further subdivisions within the other $(j+1)$ parts $\mathsf{p}_1^-, \dots, \mathsf{p}_j^-, \mathsf{p}_{j+1}^-$ of $\mathbb{R}^d$ in the time interval

(s_{j+1}, s_{j+2}). Consequently, for the length-weighted typical maximal segment $\mathsf{MP}_{1,1}^{(t)}$, a corresponding Poisson distributed number of internal vertices appear, whose parameter is $(j+1)\Lambda(\langle[0,u]\rangle)x(s_{j+2}-s_{j+1})$. At time s_{d-1}, the last $(d-1)$-dimensional maximal polytope p_{d-1} is born and $\mathbb{R}^d$ is divided into d parts. During the last time interval (s_{d-1}, t), internal vertices of I could be created by further subdivisions within all these d parts of $\mathbb{R}^d$. This leads to a Poisson distribution with parameter $d \cdot \Lambda(\langle[0,u]\rangle)x(t-s_{d-1})$ for such an I, and hence for $\mathsf{MP}_{1,1}^{(t)}$. Adding all the above independent Poisson distributed numbers, we obtain a Poisson distributed number of vertices with parameter $\Lambda(\langle[0,u]\rangle)x(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)$ in the relative interior of $\mathsf{MP}_{1,1}^{(t)}$ – the length-weighted maximal segment of the STIT tessellation $Y(t)$. Formally, this means that

$$\mathbb{P}\big(N = n | \ell = x, D = u, \beta_1 = s_1, \ldots, \beta_{d-1} = s_{d-1}\big)$$
$$= \frac{\big[\Lambda(\langle[0,u]\rangle)x(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)\big]^n}{n!} e^{-\Lambda(\langle[0,u]\rangle)x(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)}.$$

Thus,

$$\begin{aligned}\mathsf{p}_{1,1}(n) = \int_0^t \int_0^{s_{d-1}} \cdots \int_0^{s_2} \int_{\mathcal{S}_+^{d-1}} \int_0^\infty & \frac{(d-1)!}{t^{d-1}} [\Lambda(\langle[0,u]\rangle)]^2 s_{d-1}^2 x e^{-\Lambda(\langle[0,u]\rangle)s_{d-1}x} \times \\ & \times \frac{\big[\Lambda(\langle[0,u]\rangle)x(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)\big]^n}{n!} \times \\ & \times e^{-\Lambda(\langle[0,u]\rangle)x(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)}\, \mathrm{d}x\, \mathbb{P}_D(\mathrm{d}u)\, \mathrm{d}s_1 \ldots \mathrm{d}s_{d-2}\mathrm{d}s_{d-1}.\end{aligned} \tag{56}$$

Integrating now with respect to x, we first observe that

$$\int_0^\infty (\Lambda(\langle[0,u]\rangle)x)^{n+1} e^{-\Lambda(\langle[0,u]\rangle)s_{d-1}x} e^{-\Lambda(\langle[0,u]\rangle)x(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)}\, \mathrm{d}x$$
$$= \frac{(n+1)!}{\Lambda(\langle[0,u]\rangle)(d\cdot t - s_{d-1} - s_{d-2} - \ldots - s_1)^{n+2}}.$$

Therefore, Equation (56) can be transformed into

$$\mathsf{p}_{1,1}(n) = (n+1)(d-1)! \int_0^t \int_0^{s_{d-1}} \cdots \int_0^{s_2} \frac{s_{d-1}^2}{t^{d-1}} \frac{(d\cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)^n}{(d\cdot t - s_{d-1} - s_{d-2} - \ldots - s_1)^{n+2}}\, \mathrm{d}s_1 \ldots \mathrm{d}s_{d-2}\mathrm{d}s_{d-1}.$$

The value of $\mathsf{p}_{1,1}(n)$ for $Y(t)$ is the same as that for $tY(t)$ since the number of internal vertices does not change when the tessellation is rescaled. Because of (20), the value of $\mathsf{p}_{1,1}(n)$ for $tY(t)$, however, is the same as that for $Y(1)$. We infer that $\mathsf{p}_{1,1}(n)$ is independent of t.

The formula for the mean number of internal vertices of the length-weighted typical maximal segment of $Y(t)$ can be obtained from $\sum\limits_{n=0}^{\infty} n\, \mathsf{p}_{1,1}(n)$ by interchanging summation with integration and then by a straight forward calculation as follows

$$\mathbb{E}N = \sum_{n=0}^{\infty} n\,\mathsf{p}_{1,1}(n) = \frac{(d-1)!}{t^{d-1}} \int_0^t \int_0^{s_{d-1}} \dots \int_0^{s_2} s_{d-1}^2 \sum_{n=0}^{\infty} n(n+1) \times$$
$$\times \frac{(d\cdot t - 2s_{d-1} - s_{d-2} - \dots - s_1)^n}{(d\cdot t - s_{d-1} - s_{d-2} - \dots - s_1)^{n+2}} \mathrm{d}s_1 \dots \mathrm{d}s_{d-2}\mathrm{d}s_{d-1}$$
$$= \frac{2(d-1)!}{t^{d-1}} \int_0^t \int_0^{s_{d-1}} \dots \int_0^{s_2} \frac{d\cdot t - 2s_{d-1} - s_{d-2} - \dots - s_1}{s_{d-1}} \mathrm{d}s_1 \dots \mathrm{d}s_{d-2}\mathrm{d}s_{d-1}$$
$$= \frac{2(d-1)!}{t^{d-1}} \int_0^t \left(\frac{d\cdot t s_{d-1}^{d-3}}{(d-2)!} - \frac{(d+2)s_{d-1}^{d-2}}{2(d-2)!} \right) \mathrm{d}s_{d-1}.$$

If $d=2$ then $\mathbb{E}N=\infty$. If $d\geq 3$ then $\mathbb{E}N = (d^2-2d+4)/(d-2)$. To conclude, the argument for the typical maximal segment is similar. In fact, we use Lemmas 4.1.8 and 4.1.10, [**8**, Theorem 1] and the Interval Theorem in [**9**, Page 39] in that order to get

$$p_{\ell(\mathsf{MP}_{1,0}^{(t)})|D(\mathsf{MP}_{1,0}^{(t)})=u,\beta_1(\mathsf{MP}_{1,0}^{(t)})=s_1,\dots,\beta_{d-1}(\mathsf{MP}_{1,0}^{(t)})=s_{d-1}}(x)$$
$$= p_{\ell(\mathsf{MP}_{1,0}^{(t)})|D(\mathsf{MP}_{1,0}^{(t)})=u,\beta_{d-1}(\mathsf{MP}_{1,0}^{(t)})=s_{d-1}}(x)$$
$$= p_{\ell(\mathsf{F}_{1,0}^{(s_{d-1})})|D(\mathsf{F}_{1,0}^{(s_{d-1})})=u}(x) = \Lambda(\langle[0,u]\rangle)s_{d-1}e^{-\Lambda(\langle[0,u]\rangle)s_{d-1}x}.$$

Moreover, the fact that $D(\mathsf{MP}_{1,0}^{(t)})$ is independent of the birth-time vector $(\beta_1(\mathsf{MP}_{1,0}^{(t)}), \dots, \beta_{d-1}(\mathsf{MP}_{1,0}^{(t)}))$ is shown with a similar method as for the length-weighted case. Applying Theorem 4.2.1 for the case $k=1$ and $j=0$, we obtain

$$p_{\beta_1(\mathsf{MP}_{1,0}^{(t)}),\dots,\beta_{d-1}(\mathsf{MP}_{1,0}^{(t)})|D(\mathsf{MP}_{1,0}^{(t)})=u}(s_1,\dots,s_{d-1})$$
$$= p_{\beta_1(\mathsf{MP}_{1,0}^{(t)}),\dots,\beta_{d-1}(\mathsf{MP}_{1,0}^{(t)})}(s_1,\dots,s_{d-1}) = d(d-2)!\frac{s_{d-1}}{t^d}.$$

$\square$

Remark 4.3.2. In the planar case $d=2$, $\mathsf{p}_{1,0}(n)$ is known from [**33, 16**], whereas for $d=3$ the formula for $\mathsf{p}_{1,0}(n)$ has been established in [**36**] by different methods. Our approach is more general and allows to deduce the corresponding formula for the length-weighted maximal segment as well as to deal with arbitrary space dimensions. To provide a concrete example, take $d=3$ and consider the length-weighted typical maximal segment. Here, we have

$$\mathsf{p}_{1,1}(0) = 5 + 18\ln 2 - \frac{63}{4}\ln 3 \approx 0.173506\,,$$
$$\mathsf{p}_{1,1}(1) = 28 + 90\ln 2 - \frac{657}{8}\ln 3 \approx 0.159712, \qquad \text{etc.}$$

The mean number of internal vertices equals 7 in this case. The values $\mathsf{p}_{1,1}(n)$ may be determined from the formula in Theorem 4.3.1 by a straightforward integration (or with computer assistance).

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